7. Natural limits on observations

He who asks a question is a fool for five minutes;
he who does not ask a question remains a fool forever.
Chinese Proverb

7.1 Limits on differentiation: scale, accuracy and order

For a given order of differentiation we find that there is a limiting scale-size below which the results are no longer exact. E.g. when we study the derivative of a ramp with slope 1, we expect the outcome to be correct. Let us look at the *observed* derivative at the center of the image for a range of scales ($0.4 < \sigma < 1.2$ in steps of 0.1):

```
<< FrontEndVision`FEV`; im = Table[x, {y, 64}, {x, 64}];
b = Table[{σ, gDf[im, 1, 0, σ][[32, 32]]}, {σ, .4, 1.2, .1}];
ListPlot[b, PlotJoined → True,
  PlotRange → All, AxesLabel → {"σ", "∂xL"},
  PlotStyle -> Thickness[.01], ImageSize -> 250];
```

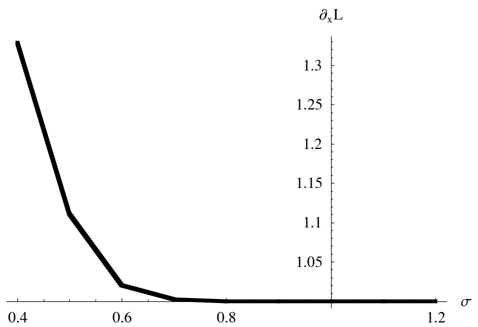

Figure 7.1 The measured derivative value of the function $y = x$ is no longer correct for decreasing scale. For scales $\sigma < 0.6$ pixels a marked deviation occurs.

The value of the derivative starts to deviate for scales smaller then say $\sigma = 0.6$. Intuitively, we understand that something must go wrong, when we decrease the size of the kernel in the spatial domain: it becomes increasingly difficult to fit the Gaussian derivative function with its zerocrossings. We recall from chapter 4 that the number of zerocrossings of a Gaussian derivative kernel is equal to the order of differentiation.
There is a fundamental relation between the order of differentiation, scale of the operator and the accuracy required [TerHaarRomeny1994b]. We will derive now this relation.
The Fourier transform of a Gaussian kernel is again a Gaussian:

```
Unprotect[gauss]; gauss[x_, σ_] := 1/Sqrt[2 π σ^2] Exp[-x^2/(2 σ^2)];
fftgauss[ω_, σ_] = FourierTransform[gauss[x, σ], x, ω]
```

$$\frac{e^{-\frac{1}{2}\sigma^2\omega^2}}{\sqrt{2\pi}}$$

The Fourier transform of the derivative of a function is $-i\,\omega$ times the Fourier transform of the function:

```
FourierTransform[∂x gauss[x, σ], x, ω] / FourierTransform[gauss[x, σ], x, ω]
```

$$-i\,\omega$$

The Fourier transform of the n-th derivative of a function is $(-i\,\omega)^n$ times the Fourier transform of the function. Note that there are several definitions for the signs (see the *Mathematica* Help browser for **Fourier**).
A smaller kernel in the spatial domain gives rise to a wider kernel in the Fourier domain, as shown below for a range of widths of first order derivative Gaussian kernels (in 1D):

```
DisplayTogetherArray[
  {Plot3D[fftgauss[ω, σ], {ω, -π, π}, {σ, .4, 2}, PlotPoints → 30,
    AxesLabel → {"ω", "σ", "fft"}, Axes → True, Boxed → True],
   Plot3D[gauss[x, σ], {x, -π, π}, {σ, .4, 2}, PlotPoints → 30, AxesLabel →
     {"x", "σ", "gauss"}, Axes → True, Boxed → True]}, ImageSize -> 490];
```

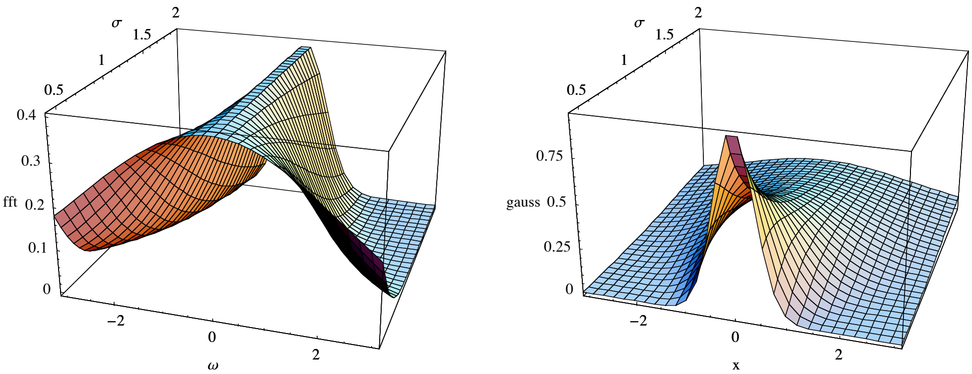

Figure 7.2 Left: The Fourier transform of the Gaussian kernel is defined for $\pi < \omega < \pi$. The function repeats forever along the frequency axis over this domain. For decreasing scale σ in the spatial domain the Fourier transform get wider in the spatial frequency domain. At some value of σ a significant leakage (aliasing) occurs. Right: The spatial Gaussian kernel as a function of scale.

We plot the Fourier spectrum of a kernel, and see that the function has signal energy outside its proper domain [-π, π] for which the spectrum is defined:

```
FilledPlot[{If[-π < ω < π, fftgauss[ω, .5], 0], fftgauss[ω, .5]},
  {ω, -2 π, 2 Pi}, Fills → {{{1, Axis}, GrayLevel[.5]}},
  Ticks → {{-π, π}, Automatic},
  AxesLabel → {"ω", "𝒢(ω,σ=.5)"}, ImageSize -> 350];
```

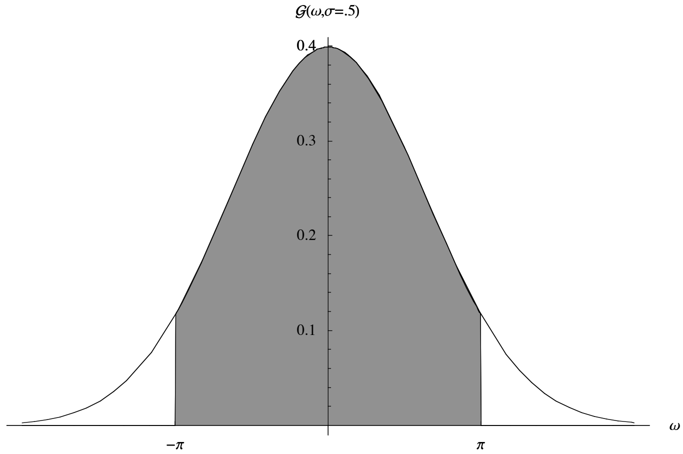

Figure 7.3 The definition of the leakage is the (unshaded) area under the curve outside the definition domain, relative to the total area under the curve. Here the definition is given for the 1D Gaussian kernel. Due to the separability of Gaussian kernels this definition is easily extended to higher dimensions.

The error is defined as the amount of the *energy* (the square) of the kernel that is 'leaking' relative to the total area under the curve (note the integration ranges):

$$\text{error[n\_, σ\_]} = 100 \frac{\int_{\pi}^{\infty} (\text{I}\,\omega)^{2\,n}\ \text{fftgauss}[\omega,\sigma]^2\, d\omega}{\int_{0}^{\infty} (\text{I}\,\omega)^{2\,n}\ \text{fftgauss}[\omega,\sigma]^2\, d\omega}$$

$$\frac{100\ ((1+2\,n)\ \text{Gamma}[\frac{1}{2}+n] - 2\ \text{Gamma}[\frac{3}{2}+n] + (1+2\,n)^2\ \text{Gamma}[\frac{1}{2}+n,\ \pi^2\,\sigma^2])}{(1+2\,n)^2\ \text{Gamma}[\frac{1}{2}+n]}$$

We plot this Gamma function for scales between $\sigma = 0.2 - 2$ and order of differentiation from 0 to 10, and we insert the 5% error line in it (we have to lower the plot somewhat (-6%) to make the line visible):

```
Block[{$DisplayFunction = Identity},
  p1 = Plot3D[error[n, σ] - 6, {σ, .2, 2}, {n, 0, 10}, PlotRange → All,
    AxesLabel → {"σ", "n", "error %"}, Boxed → True, Axes → True];
  p2 = ContourPlot[error[n, σ], {σ, .2, 2}, {n, 0, 10},
    ContourShading → False, Contours → {5}];
  c3d = Graphics3D[Graphics[p2][[1]] /.
     Line[pts_] :> ({Thickness[.01], val = Apply[error, First[pts]];
        Line[Map[Append[#, val] &, pts]]})]];
```

```
Show[p1, c3d, ImageSize -> 310];
```

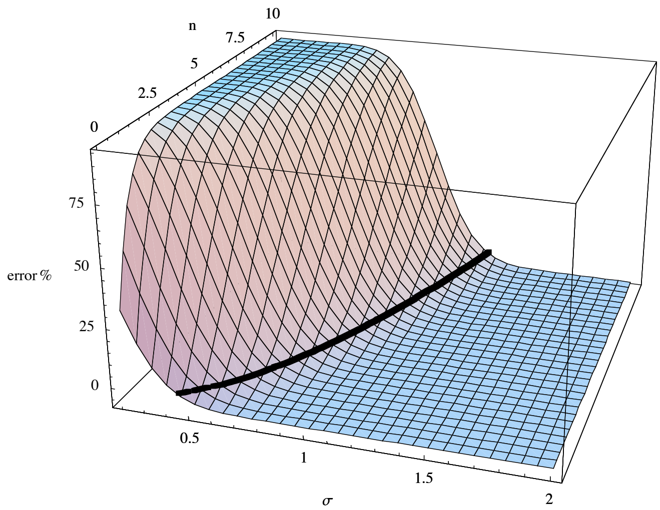

Figure 7.4 Relation between scale σ, order of differentiation n, and accepted error (in %) for a convolution with a Gaussian derivative function, implemented through the Fourier domain.

```
ContourPlot[error[n, σ], {n, 0, 10}, {σ, .1, 2},
  ContourShading → False, Contours → {1, 5, 10}, FrameLabel ->
   {"Order of differentiation", "Scale in pixels"}, ImageSize -> 275];
```

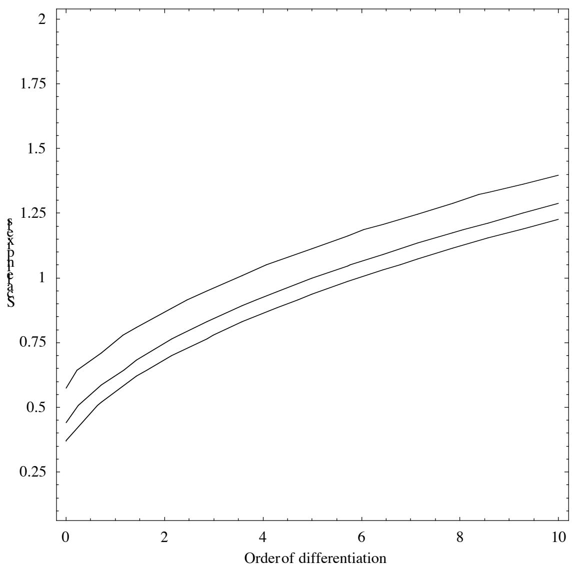

Figure 7.5 Relation between scale σ and the order of differentiation n for three fixed accuracies for a convolution with a Gaussian derivative function, implemented through the Fourier domain: upper graph: 1%, middle graph: 5%, lower graph: 10% accepted error.

The lesson from this section is that we should never make the scale of the operator, the Gaussian kernel, too small. The lower limit is indicated in the graph above. A similar reasoning can be set up for the outer scale, when the aliasing occurs in the spatial domain.

We summarize with a table of the minimum σ, given accuracies of 1, 5, resp 10%, and differentiation up to fifth order:

```
TableForm[Table[
  Prepend[(σ /. FindRoot[error[n, σ] == #, {σ, .6}]) & /@ {1, 5, 10}, n],
  {n, 1, 5}], TableHeadings ->
  {None, {"Order", "σ @ 1%", "σ @ 5%", "σ @ 10%"}}]
```

Order	σ @ 1%	σ @ 5%	σ @ 10%
1	0.758115	0.629205	0.56276
2	0.874231	0.748891	0.684046
3	0.967455	0.844186	0.78025
4	1.04767	0.925811	0.862486
5	1.11919	0.998376	0.935501

▲ Task 7.1 This chapter discusses the fundamental limit which occurs by too much 'broadening' of the Gaussian kernel in the Fourier domain for small scales (the 'inner scale limit'). Such a broadening also occurs in the spatial domain, when we make the scale too large. A similar fundamental limit can be established for the 'outer scale limit'.
Find the relation between scale, differential order and accuracy for the outer scale.

The reasoning in this chapter is based on the implementation of a convolution in the Fourier domain. The same reasoning holds however when other choices are made for the implementation. In each case, a decision about the periodicity or extension of the image values outside the domain (see the discussion in chapter 5), determines the fundamental limit discussed here.

7.2 Summary of this chapter

There is a limit to the order of differentiation for a given scale of operator and required accuracy. The limit is due to the no longer 'fitting' of the Gaussian derivative kernel in its Gaussian envelop, known as aliasing. We derived the analytic expression for this error.

As a rule of thumb, for derivatives up to 4^{th} order, the scale should be not less then one pixel.