Deblurring with a scale-space approach

Gaussian degradation occurs in a large number of situations.

- the point-spread-function of the human lens e.g. has a close to Gaussian shape (for a 3 mm pupil σ is about 2 minutes of arc);

- the atmospheric turbulence blurs astronomical images in a Gaussian fashion;

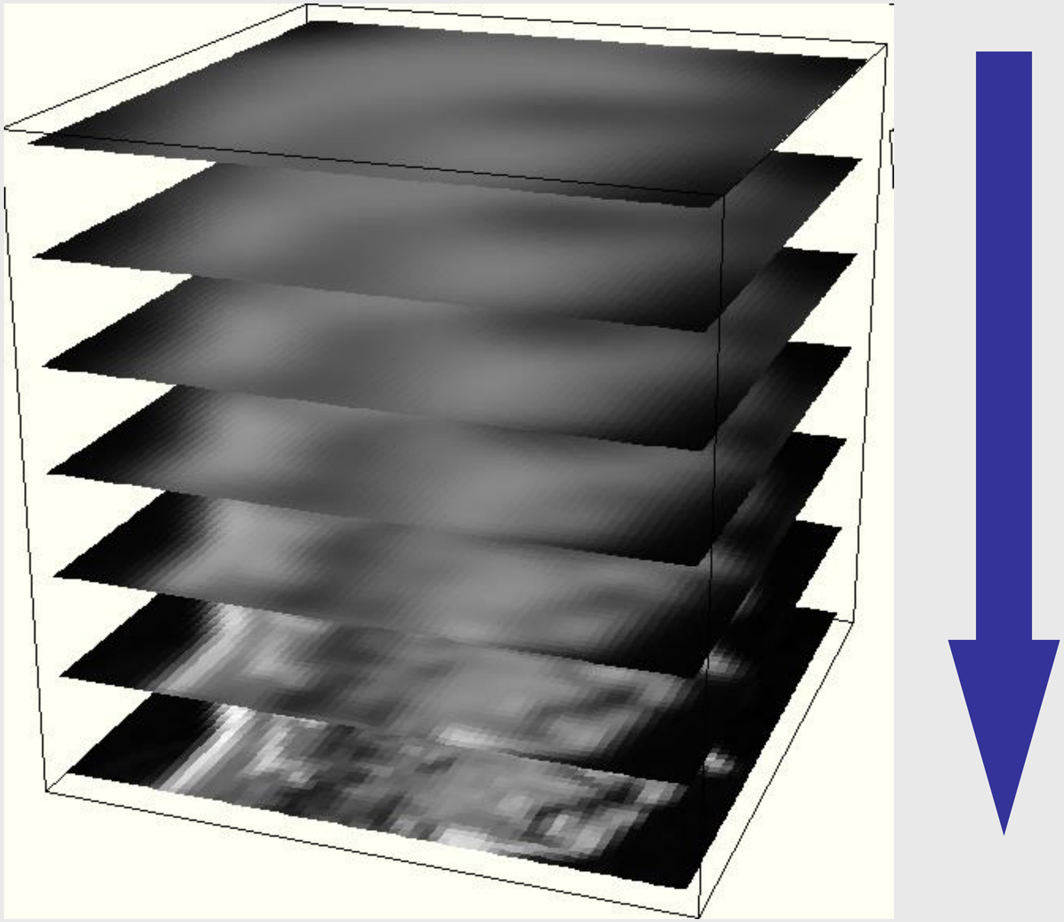
- the thickness profile of thin slices made by modern spiral-CT scanners is about Gaussian, leading to Gaussian blur in a multiplanar reconstructed image such as in the sagittal or coronal plane.

There is an analytical solution for the inversion of Gaussian blur. But the reconstruction can never be exact.

Many practical solutions have been proposed, involving a variety of enhancing filters (e.g. high-pass or Wiener) and Fourier methods.

Analytical methods have been proposed by Kimia, Hummel and Zucker [Kimia1986, Kimia1993, Hummel1987] as well as Reti [Reti 1995a]. They replaced the Gaussian blur kernel by a highly structured Toeplitz matrix and deblurred the image by the analytical inverse of this matrix. Martens deblurred images with polynomial transforms [Martens 1990].

Can we inverse the diffusion equation?



Recall that scale-space is infinitely differentiable due to the regularization properties of the observation process.

We can construct a Taylor expansion of the scale-space in any direction, including the negative scale direction.

Taylor expansion ‘downwards’:

$$L(x, y, s-\delta s) = L - \frac{\partial L}{\partial s}\,\delta s + \frac{1}{2!}\,\frac{\partial^2 L}{\partial s^2}\,\delta s^2 - \frac{1}{3!}\,\frac{\partial^3 L}{\partial s^3}\,\delta s^3 + O(\delta s)^4$$

The derivatives with respect to s (scale) can be expressed in spatial derivatives due to the diffusion equation

$$\frac{\partial L}{\partial s} = \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2}$$

$$L(x, y, s - \delta s) =$$

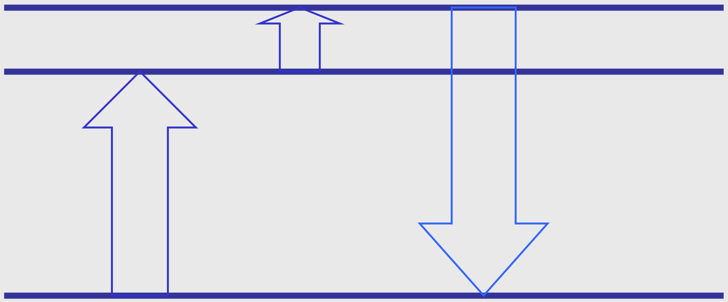
$$L - \left(\frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} \right) \delta s +$$

$$\frac{1}{2!} \left(\frac{\partial^4 L}{\partial x^4} + 2 \frac{\partial^4 L}{\partial x^2 \partial L^2} + \frac{\partial^4 L}{\partial y^4} \right) \delta s^2 - O(\delta s)^3$$

It is well-known that subtraction of the Laplacian sharpens the image. It is the first order approximation of the deblurring process.

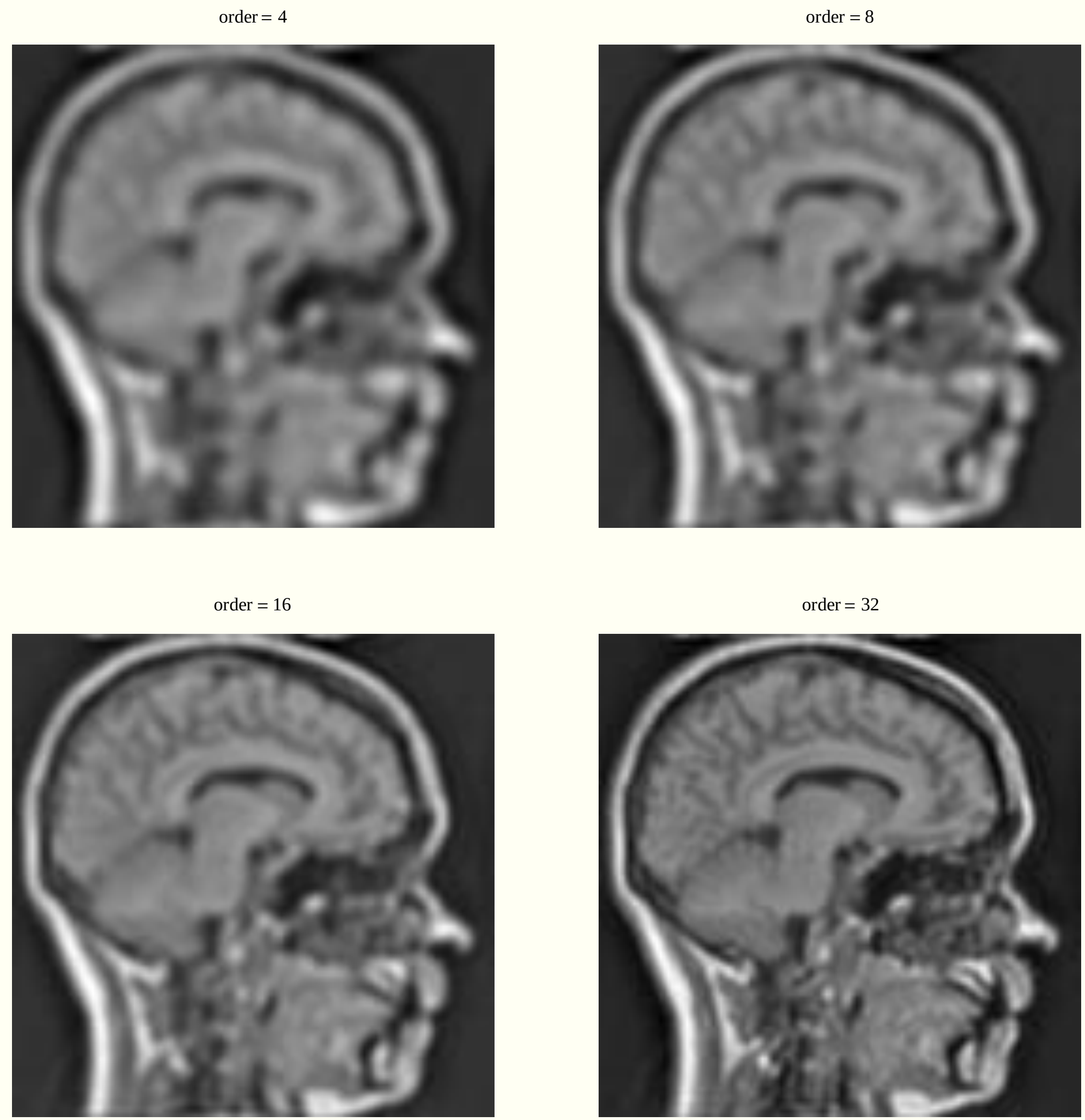
We must account for the scale of the differential operators:

$$t_{\text{deblur}} = -\frac{\sigma^2{}_{\text{estimated}} + \sigma^2{}_{\text{operator}}}{2}$$



Deblurring to 4th, 8th, 16th and 32nd order:

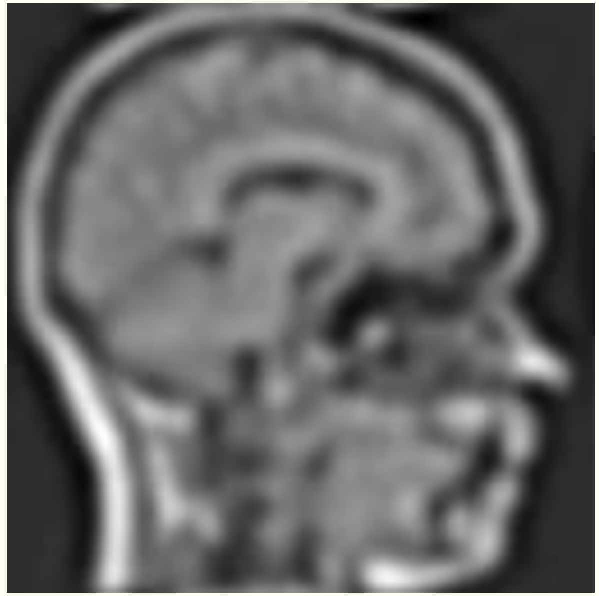
There are 560 derivative terms in the 32nd order expression!



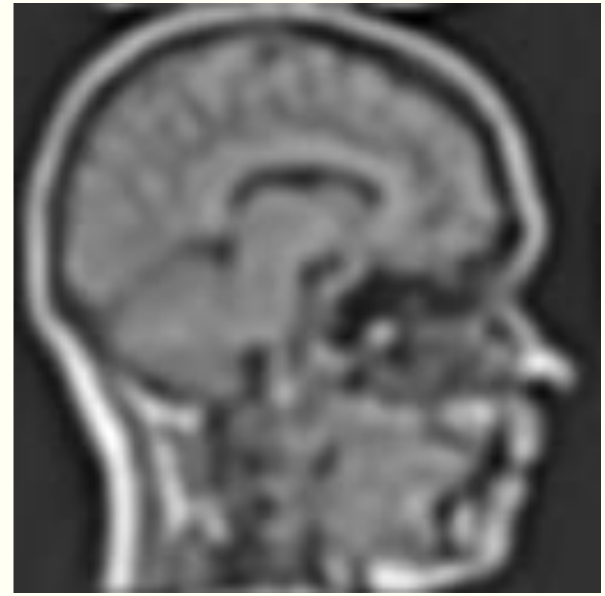
Rounding the data to 8 bit gives less information per pixel

Deblurring to 4th, 8th, 16th and 32nd order:

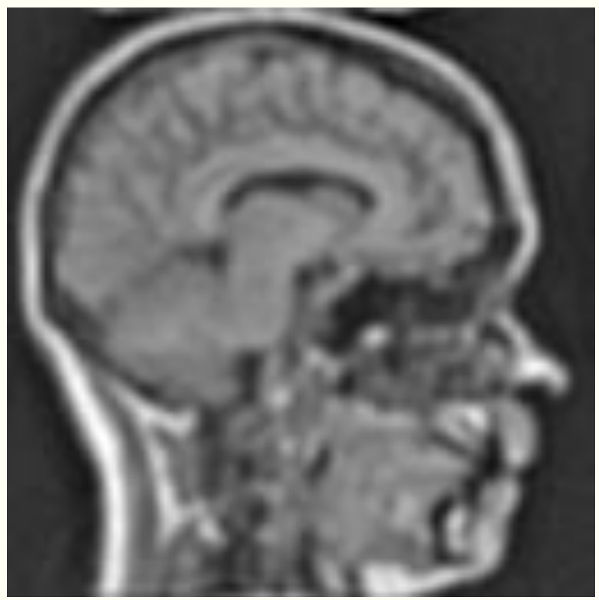
order = 4



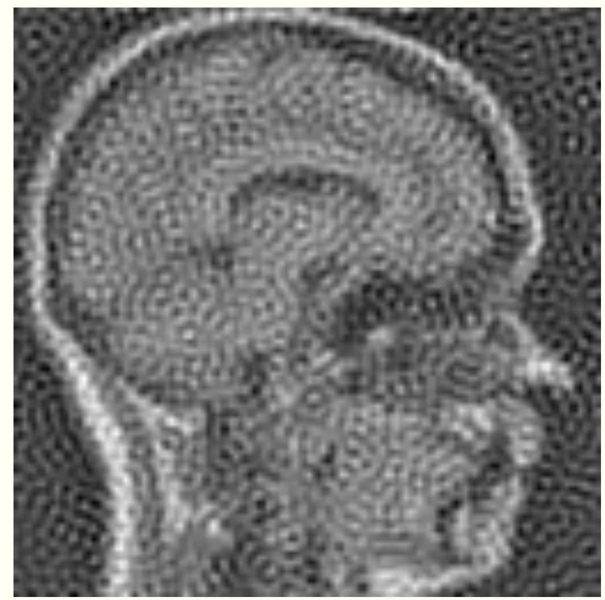
order = 8



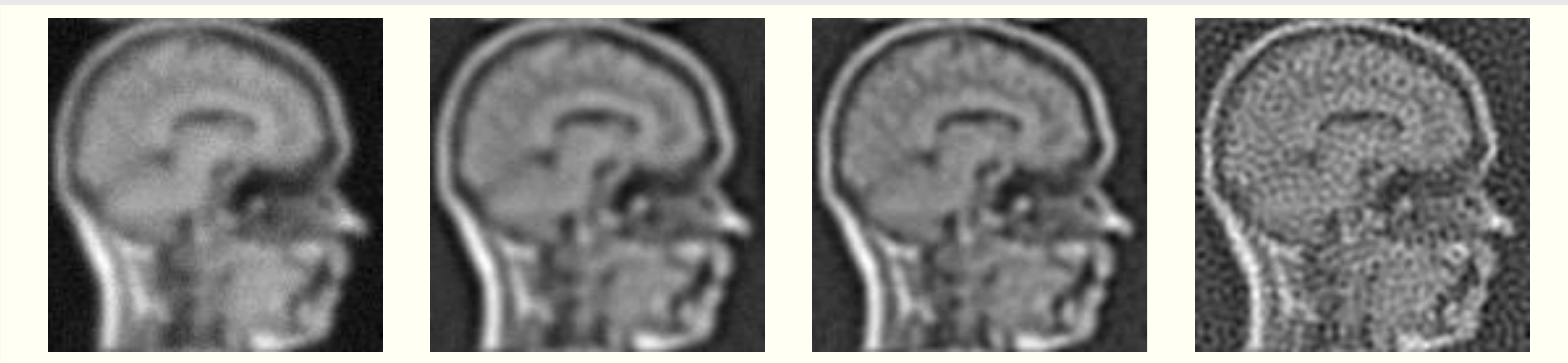
order = 16



order = 32

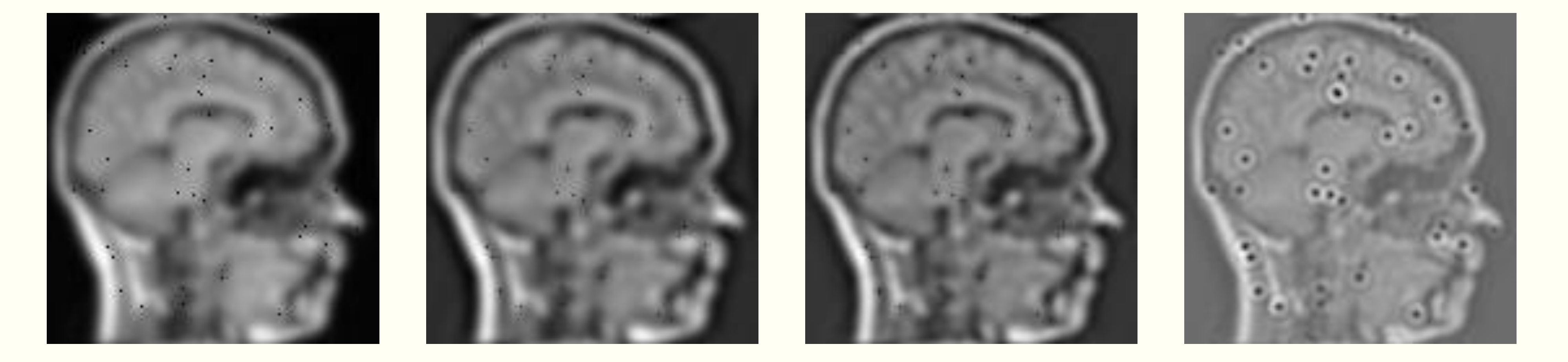


Influence of additive Gaussian noise:



Noise σ=5 order = 4 order = 8 order = 16

Holes in the image:



Shot noise order = 4 order = 8 order = 16