Beach Cities Robotics

FIRST Team 294

Multi-Linear Interpolation

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1 Abstract

A standard linear algebra method of interpolation that scales to any dimensionality is described.

2 Description

Given an array (or table) of values for a function of one or more variables, it is often desired to find a value between two given points. If the given function is not linear, then the interpolated value will be an approximation. Computation of error terms for the non-linear case is covered in standard mathematics texts [1]. If the array has more than two dimensions, the value sought will be at a point within the interior of the corresponding polytope. In the descriptions below, the functions for interpolation are assumed to be sufficiently close to linear.

2.1 Linear Interpolation

Suppose we have a nearly or exactly linear function

$$y = f(x) \tag{1}$$

and we are given two points, *A*, and *B*, as shown below in Figure 1:

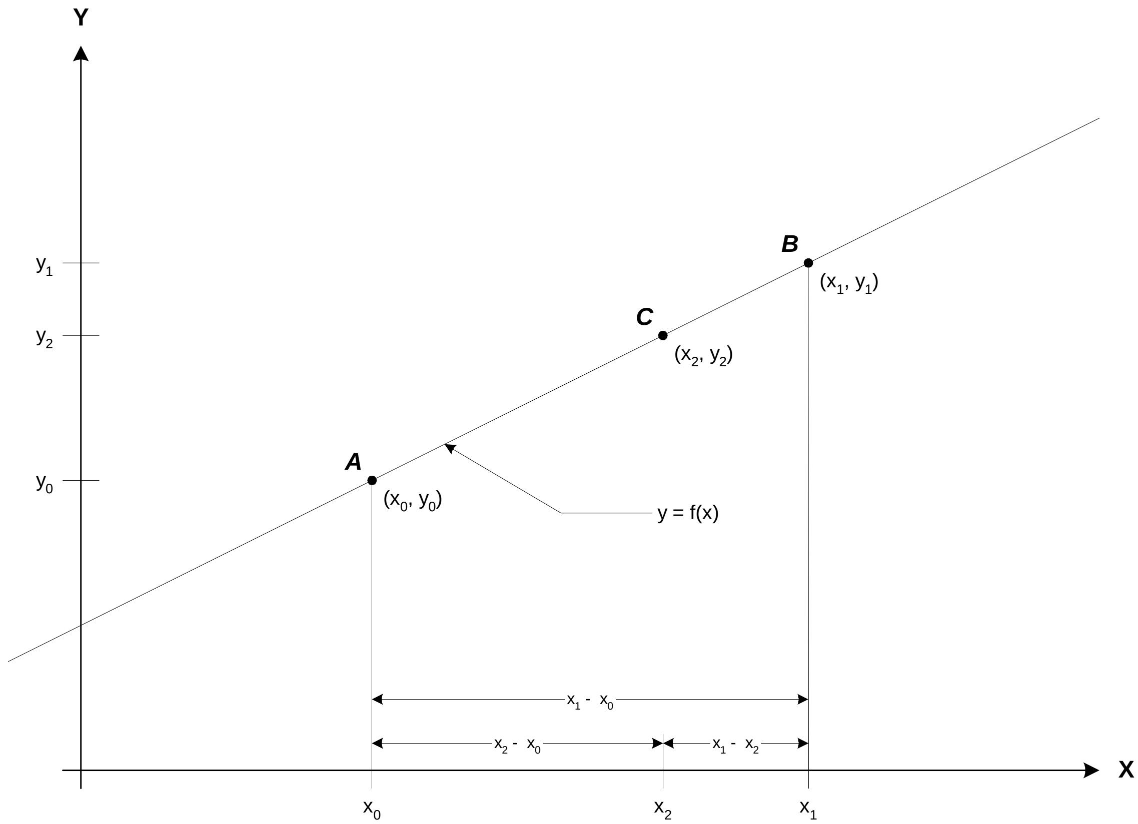

Figure 1: Linear interpolation: given points *A* and *B* and x_2, compute y_2, yielding point *C*.

To interpolate the value (y_2) of f at point C (given at x_2) we find the slope (m) of the line between points A and B. Then

$$y_2 = y_0 + m \cdot (x_2 - x_0) \tag{2}$$

where

$$m = \frac{y_1 - y_0}{x_1 - x_0} \tag{3}$$

If we expand equation (2) by substituting m from equation (3) we have

$$y_2 = \frac{y_0 \cdot (x_1 - x_0) + (y_1 - y_0) \cdot (x_2 - x_0)}{x_1 - x_0} \tag{4}$$

Performing the multiplications in equation (4) and simplifying we have

$$y_2 = \frac{y_0 \cdot (x_1 - x_2) + y_1 \cdot (x_2 - x_0)}{x_1 - x_0} \tag{5}$$

The segment of the X axis designated by $x_1 - x_0$ is the projection of segment AB onto the X axis. Dividing by this length in equation (5) normalizes the sum of the projections of segments *AC* and *CB* to a unit segment. So equation (5) is saying that the interpolated value, y_2, is y_0 times the normalized projection of segment *CB* plus y_1 times the normalized projection of segment *AC*. Note that the normalized segments that serve as factors for the *y*-values of the vertices A and B are opposite from (not adjacent to) their corresponding vertices. This interpretation of linear interpolation allows us to easily extend it to higher dimensionality.

2.2 Bilinear Interpolation

The coverage interpretation of linear interpolation is now applied to the next higher dimensionality, 3D. Bilinear interpolation is derived slightly differently in reference [2], but with identical results. Suppose we have a nearly or exactly planar function

$$z = f(x, y) \tag{6}$$

and we are given four points, A, B, C, and D defining a rectangle as shown below in Figure 2:

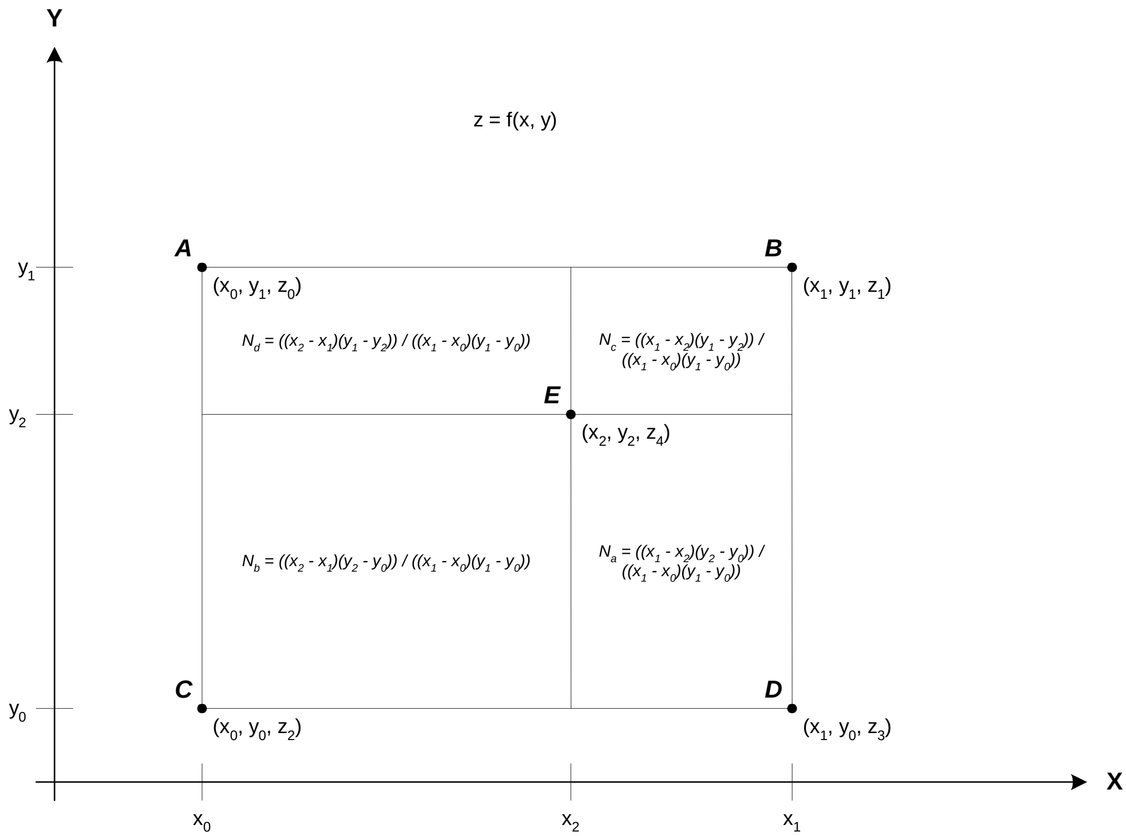

Figure 2: Bilinear interpolation. The rectangle *ABCD* is partitioned into four areas by the lines $x = x_2$ and $y = y_2$.

The rectangle *ABCD* is partitioned into four areas by the lines $x = x_2$ and $y = y_2$. To interpolate the value (z_4) of f at point E (given at x_2 and y_2) we first find the normalized areas of the partitions of the rectangle *ABCD*. The four areas are normalized by dividing them each by the area of rectangle *ABCD*.

The four normalized areas, N_a, N_b, N_c, and N_d, each diagonally opposite from its naming rectangle vertex, are given by

$$N_a = \frac{(x_1 - x_2)\cdot(y_2 - y_0)}{(x_1 - x_0)\cdot(y_1 - y_0)} \tag{7}$$

$$N_b = \frac{(x_2 - x_0)\cdot(y_2 - y_0)}{(x_1 - x_0)\cdot(y_1 - y_0)} \tag{8}$$

$$N_c = \frac{(x_1 - x_2)\cdot(y_1 - y_2)}{(x_1 - x_0)\cdot(y_1 - y_0)} \tag{9}$$

$$N_d = \frac{(x_2 - x_0)\cdot(y_1 - y_2)}{(x_1 - x_0)\cdot(y_1 - y_0)} \tag{10}$$

Then z_4 is computed by the equation

$$z_4 = z_0 \cdot N_a + z_1 \cdot N_b + z_2 \cdot N_c + z_3 \cdot N_d \tag{11}$$

2.3 Trilinear Interpolation

Suppose we have a nearly or exactly linear function

$$v = f(x, y, z) \tag{12}$$

and we are given eight points, *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H* defining a right rectangular prism as shown below in Figure 3:

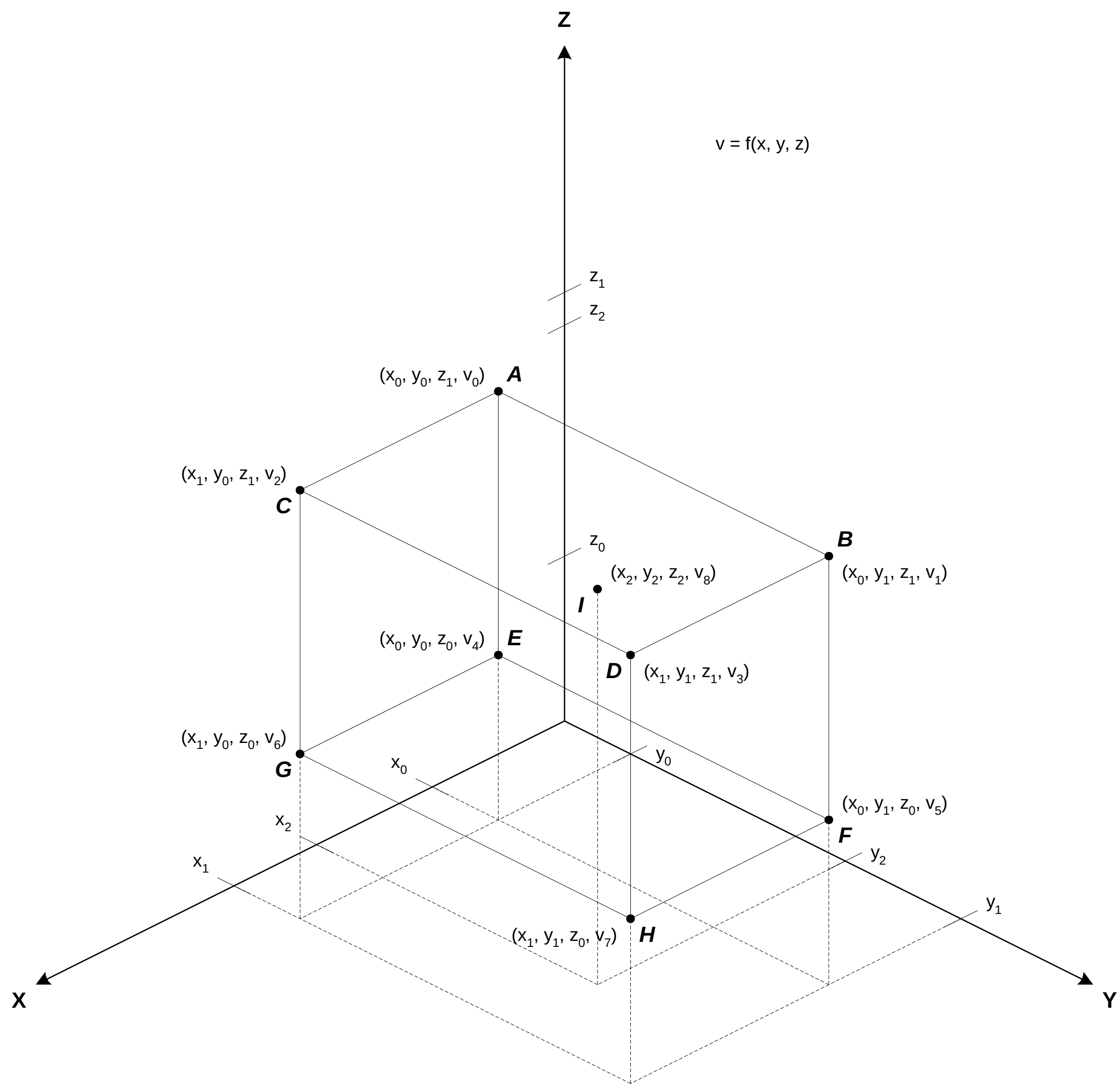

Figure 3: Trilinear interpolation. The right rectangular prism *ABCDEFGH* is partitioned into eight volumes by the three orthogonal planes passing through point *I*.

The prism *ABCDEFGH* is partitioned into eight volumes by the planes $x = x_2$, $y = y_2$, and $z = z_2$ To interpolate the value (v_8) of f at point I (given at x_2, y_2, and z_2) we first find the normalized volumes of the partitions of the prism *ABCDEFGH*. The eight volumes are normalized by dividing them each by the volume of prism *ABCDEFGH*.

The eight normalized volumes N_a, N_b, N_c, N_d N_e, N_f, N_g, and N_h, each diagonally opposite from its naming prism vertex, are given by

$$N_a = \frac{(x_1 - x_2)\cdot(y_1 - y_2)\cdot(z_2 - z_0)}{(x_1 - x_0)\cdot(y_1 - y_0)\cdot(z_1 - z_0)} \tag{13}$$

$$N_b = \frac{(x_1 - x_2)\cdot(y_2 - y_0)\cdot(z_2 - z_0)}{(x_1 - x_0)\cdot(y_1 - y_0)\cdot(z_1 - z_0)} \tag{14}$$

$$N_c = \frac{(x_2 - x_0)\cdot(y_1 - y_2)\cdot(z_2 - z_0)}{(x_1 - x_0)\cdot(y_1 - y_0)\cdot(z_1 - z_0)} \tag{15}$$

$$N_d = \frac{(x_2 - x_0)\cdot(y_2 - y_0)\cdot(z_2 - z_0)}{(x_1 - x_0)\cdot(y_1 - y_0)\cdot(z_1 - z_0)} \tag{16}$$

$$N_e = \frac{(x_1 - x_2)\cdot(y_1 - y_2)\cdot(z_1 - z_2)}{(x_1 - x_0)\cdot(y_1 - y_0)\cdot(z_1 - z_0)} \tag{17}$$

$$N_f = \frac{(x_1 - x_2)\cdot(y_2 - y_0)\cdot(z_1 - z_2)}{(x_1 - x_0)\cdot(y_1 - y_0)\cdot(z_1 - z_0)} \tag{18}$$

$$N_g = \frac{(x_2 - x_0)\cdot(y_1 - y_2)\cdot(z_1 - z_2)}{(x_1 - x_0)\cdot(y_1 - y_0)\cdot(z_1 - z_0)} \tag{19}$$

$$N_h = \frac{(x_2 - x_0)\cdot(y_2 - y_0)\cdot(z_1 - z_2)}{(x_1 - x_0)\cdot(y_1 - y_0)\cdot(z_1 - z_0)} \tag{20}$$

Then z_8 is computed by the equation

$$z_8 = z_0 \cdot N_a + z_1 \cdot N_b + z_2 \cdot N_c + z_3 \cdot N_d + z_4 \cdot N_e + z_5 \cdot N_f + z_6 \cdot N_g + z_7 \cdot N_h \tag{21}$$

2.4 General Multi-Linear Interpolation

We lose our geometric intuition for linear interpolation in higher dimensions. However, the same principles apply. To make the pattern evident in our generalized equations, we no longer designate the vertices of polytopes by letters and the corresponding diagonally opposite normalized partition polytopes are designated by subscripts that show the partition senses for the partitioned dimensions.

Suppose we have a nearly or exactly linear function

$$y = f(x_1, x_2, \ldots x_n) \tag{22}$$

and we are given 2^n points that are the vertices of a right rectangular polytope. If we are given a point $(x_{12}, x_{22}, \ldots, x_{n2})$ in the interior of the polytope, the polytope is divided into 2^n partitions by n m-dimensional branes passing through the given interior point where $m = n - 1$.

Each partition has a normalized quantity given by

$$N_{00\ldots0} = \frac{(x_{11} - x_{12})\cdot(x_{21} - x_{22})\cdot\ldots\cdot(x_{n1} - x_{n2})}{(x_{11} - x_{10})\cdot(x_{21} - x_{20})\cdot\ldots\cdot(x_{n1} - x_{n0})} \tag{23}$$

$$N_{00\ldots1} = \frac{(x_{11} - x_{12})\cdot(x_{21} - x_{22})\cdot\ldots\cdot(x_{n2} - x_{n0})}{(x_{11} - x_{10})\cdot(x_{21} - x_{20})\cdot\ldots\cdot(x_{n1} - x_{n0})}$$

$$\vdots$$

$$N_{11\ldots1} = \frac{\left(x_{12} - x_{10}\right)\cdot\left(x_{22} - x_{20}\right)\cdot\ldots\cdot\left(x_{n2} - x_{n0}\right)}{\left(x_{11} - x_{10}\right)\cdot\left(x_{21} - x_{20}\right)\cdot\ldots\cdot\left(x_{n1} - x_{n0}\right)}$$

Where point $\left(x_{10}, x_{20}, \ldots, x_{n0}\right)$ is the polytope vertex nearest the origin and point $\left(x_{11}, x_{21}, \ldots, x_{n1}\right)$ is the vertex farthest from the origin. The y-value of the point in the interior of the polytope is given by

$$y = x_1 \cdot N_1 + x_2 \cdot N_2 + \ldots + x_n \cdot N_n \tag{24}$$

3 Conclusion

A generic *n*-dimensional linear interpolation method has been described with specific low-dimensional examples.

4 References

[1] Kreysig, Erwin, *Advanced Engineering Mathematics, Third Edition*, John Wiley and Sons, Inc., 1972.

[2] Safro, Ilya, "Resampling Using Bilinear Interpolation," Web page: http://www.wisdom.weizmann.ac.il/~maksimf/ex5/Resample.html, 2004.