A Comparison of the Deep Structure of $\alpha$-Scale Spaces

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Abstract. We compare the topology and deep structure of alternative scale space representations, so called $\alpha$-scale spaces, $1/2 \leq \alpha \leq 1$, which are subject to a first order pseudo partial differential equation on the upper half plane $\{(x, s) \in \mathbb{R}^d \times \mathbb{R} \mid s > 0\}$. In particular, the cases $\alpha = 1$ and $\alpha = 1/2$, which correspond to respectively Poisson scale space and Gaussian scale space, are considered. Poisson scale space is equivalent to harmonic extension to the upper half plane, inducing potential physics, whereas Gaussian scale space is generated by the diffusion equation on the upper half plane, inducing heat physics. Despite the continuous connection (by parameter $1/2 \leq \alpha \leq 1$) between these scale spaces and the similarity between their convolution convolution kernels, we show both theoretically and experimentally that there is a strong difference between the topology in the deep structure of these scale spaces.

Keywords: $\alpha$-Scale Spaces, Deep structure, Morse Theory.

1 Introduction

In linear scale space theory one obtains a so-called $\alpha$-scale space representation $u^\alpha_f : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ of a grey value image image $f \in L^2(\mathbb{R}^d)$ by means of a holomorphic semi group generated by $-(-\Delta)^\alpha$, $0 < \alpha \leq 1$, i.e. they satisfy the unique solutions of the pseudo differential evolution system

$$
\begin{cases}
u_s = -(-\Delta)^\alpha u \\
\lim_{s \downarrow 0} u(\cdot, s) = f(\cdot)
\end{cases}, \quad (1)
$$

The unique solutions of which are obtained by means of a convolution

$$u^\alpha_f(x, s) = (K^\alpha_s * f)(x), s > 0, x \in \mathbb{R}^d, \quad (2)$$

where $K^\alpha_s = \mathcal{F}^{-1}[\omega \mapsto e^{-s\|\omega\|^{2\alpha}}]$. These isotropic linear scale space representations follow from a list of fundamental axioms, cf.[1] and the most common

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cases are $\alpha = 1$ and $\alpha = \frac{1}{2}$ leading to respectively a diffusion system and $\alpha = \frac{1}{2}$ a potential problem on the upper space $s > 0$. In these cases the convolution kernel\(^1\) equals respectively the Gaussian kernel and the Poisson kernel:

$$K_s^1(\mathbf{x}) = \frac{1}{(4\pi s)^{d/2}} e^{-\frac{\|\mathbf{x}\|^2}{4s}} \quad \text{and} \quad K_s^\frac{1}{2}(\mathbf{x}) = \frac{2}{\sigma_{d+1}} \frac{s}{(s^2 + \|\mathbf{x}\|^2)^{\frac{d+1}{2}}}.$$  

(3)

With respect to Poisson scale space case we notice that the Laplacian factorizes in two important ways:

$$\Delta_{d+1} = \frac{\partial^2}{\partial s^2} + \Delta_d = \left( \frac{\partial}{\partial s} - \sqrt{-\Delta_d} \right) \left( \frac{\partial}{\partial s} + \sqrt{-\Delta_d} \right)$$  

(4)

From this factorization it directly follows that (under the extra condition that $u(\cdot, s) \to 0$ uniformly as $s \to \infty$), that the case $\alpha = 1/2$ corresponds to harmonic extension to the upper plane, where we notice that $\left( \frac{\partial}{\partial s} + \sqrt{-\Delta_d} \right) u = 0 \iff u_s = -\sqrt{-\Delta_d} u$. Furthermore, we notice that $\alpha$-scale spaces correspond to symmetric $\alpha$-stable Levy-processes, that arise in the generalization of the Central Limit Theorem\(^2\), [3] Chapter IX, 9. 

An important geometrical quantity is the grey-value flow within an $\alpha$ scale space $u_f^\alpha$ of image $f$. This multi-scale vector field is given by

$$\mathbf{F}_\alpha[u_f^\alpha](\mathbf{x}, s) = (\mathbf{f}_s^\alpha * f)(\mathbf{x}),$$  

(5)

where $\mathbf{f}_s^\alpha(\mathbf{x}) = \mathcal{F}^{-1}[\omega \mapsto \frac{1}{\|\omega\|^{2(1-\alpha)}} \omega e^{-s\|\omega\|^2}]$. To this end we notice that

$$\frac{\partial}{\partial s}[u_f^\alpha] = -(-\Delta)^\alpha u_f^\alpha = \text{div} \mathbf{F}_\alpha[u_f],$$

which is easily verified in the Fourier domain: $-\|\omega\|^{2\alpha} = i\omega \cdot i\frac{1}{\|\omega\|^{2(1-\alpha)}} \omega$. The grey-value flow tells us how the grey-value particles flow within the scale space representation and reveals the interaction between extremal paths in scale space. For the special case of a Gaussian scale space $\alpha = 1$ the grey-value flow is obtained by means of the gradient as we have

$$\mathbf{F}_{\alpha=1}[u_f](\mathbf{x}, s) = \nabla_x u_f(\mathbf{x}, s)$$

and $\mathbf{f}_{s=1} = \nabla_x K_s^1(\mathbf{x})$. For the special case of a Poisson scale space $\alpha = \frac{1}{2}$ the grey value flow is obtained by means of the Riesz transform

$$\mathbf{F}_{\alpha=\frac{1}{2}}[u_f](\mathbf{x}, s) = \mathbf{R}_x u_f(\mathbf{x}, s)$$

\(^1\) For the other $\alpha \in (0, 1], \alpha \neq \frac{1}{2}, 1$ there do not exist closed form expressions in the spatial domain. Nevertheless, as is shown by Kanters et al.,\(^2\) $\alpha$-kernels, with $\alpha \in [\frac{1}{2}, 1]$, can accurately be approximated by convex combinations of the Poisson and Gaussian kernel. 

\(^2\) In the central limit theorem sums of identically distributed (with finite variance) independent variables are considered. If the finite variance assumption is omitted the limiting distributions become $\alpha$-kernel distributed.
and \( f_\alpha^{\frac{1}{2}} \) equals the vector-valued conjugate Poisson kernel:

\[
f_\alpha^{\frac{1}{2}}(x) = R_x K_\alpha^{1/2}(x) = Q_\alpha(x) = \frac{2}{\sigma_{d+1}} \frac{x}{(s^2 + \|x\|^2)^{d+1/2}}.
\]

By extending a scale space with its flow, one obtains a vector scale space which equals the first order jet of a Gaussian scale space if \( \alpha = 1 \) and which equals the monogenic scale space, cf.\[4\], if \( \alpha = \frac{1}{2} \), which is most practical as it comes to phase based image processing. If one considers \( \alpha \)-scale spaces on a bounded domain \( \Omega \) with reflective Neumann boundary conditions \( \frac{\partial u}{\partial n}\big|_{\partial \Omega} = 0 \) (preferable over other boundary conditions, cf. [5]) then the connection becomes even more straight forward, as in this case the generators extend to a compact self-adjoint operator on \( L^2(\Omega) \). From this observation it follows that the (generators of the) \( \alpha \)-scale spaces have a common orthonormal basis of eigen functions and the solutions are given by

\[
u_\alpha^{f,\Omega} = \sum_{m,n} f_{mn} e^{-(-\lambda_{mn})^\alpha s},
\]

where \( \Delta f_{mn} = \lambda_{mn} f_{mn} \) and implementation of the bounded domain \( \alpha \)-scale spaces for the special cases of the rectangle \([0,a] \times [0,b], a, b > 0\) and the disk \( B_{a,a}, a > 0 \) we refer to [5, 6], where also (applications of) the Monogenic scale space on a bounded domain is considered. In the limiting case where the bounded domain fills the whole \( \mathbb{R}^2 \) the solutions \([6]\) converge to convolutions with \( \alpha \)-kernels. In the deep structure differences between the bounded and unbounded domain cases only show up at either high scales or close to the boundary at lower scales. In this article we will mainly consider \( \alpha \)-scale spaces on a unbounded domain as they are much more suitable for local analysis on topology.

On the one hand the strong connection, see figure 1, between \( \alpha \)-scale spaces is interesting as it smoothly relates methods using a Gaussian scale space, to methods using a Poisson scale space and visa versa. On the other hand the strong connection/similarity on (both the unbounded domain and bounded domain) \( \alpha \)-scale spaces puts the question whether there is a relevant difference between

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Fig. 1. A short overview of the correspondence between Gaussian scale space and Poisson scale space
them. From the practical field in image analysis, there is a slight indication that Poisson scale space (and its monogenic extension) is the optimal choice for phase based processing and texture analysis, but until now an in-depth well-founded comparison has not been made.

In this article we approach this issue from a purely topological point of view, i.e. we investigate the differences (and analogies) between the deep structure of \( \alpha \)-scale spaces, which appears to be rather different. In Section 2, we give a brief introduction to the deep-structure of \( \alpha \)-scale spaces and show experimentally that the extremal curves continuously depend on \( \alpha \), if the scale in a \( \alpha \)-scale space is re-scaled by \( s \mapsto s^{\frac{1}{\alpha}} \). Nevertheless, it quite often happens in images (provided they do not only include low frequencies) that in a Poisson scale space different extrema and saddles annihilate than extrema and saddles in its Gaussian counterpart. We give a non-artificial example of an image whose Gaussian scale space contains two creations, whereas its Poisson scale space contains no creations and similarly show the extremal paths in the rescaled \( \alpha \)-scale spaces evolve as a function of \( \alpha \in \left[ \frac{1}{2}, 1 \right] \). In Section 3 we both show theoretically and experimentally that creation events do occur in a 1D-Poisson scale space, whereas they do not occur in a 1D-Gaussian scale space. In Section we investigate the difference between \( \alpha \)-scale spaces concerning causality, maximum principle and Koenderink’s principle. In a Poisson scale space maxima can increase in value over scale, which is not possible in a Gaussian scale space.

2 Deep Structure

The topological structure in a scale space and in particular the change of topological structure of \( u(\cdot, s) \) over \( s > 0 \), reflects the hierarchical structure of objects (like blobs) in an image. As the resolution increases extrema disappear until at finite scale \( S > 0 \) only one extremum is left, cf. [7]. Points in scale space where a saddle and extremum annihilate or points where an extremum and a saddle are created are called top-points. The set of top-points is given by

\[ \{(x, s) \mid (\det H_x u(\cdot, s))(x) = 0 \text{ and } (\nabla x u(\cdot, s))(x) = 0\}. \]

At these points the topological structure changes. Other interesting points in scale space are scale space saddles, these are exactly those points were \( \nabla x, s u(x, s) = (0, 0) \). Although it is possible to construct a hierarchical tree-structure by means of these points , cf. [5].

The tangent vector \( \partial_\beta(x(\beta), s(\beta)) \), with \( s = \beta \det H_x u \) of a critical path (moving with infinite speed through a top-point) in an alpha scale space \( \alpha \in (0, 1] \) is given by

\[ \partial_\beta(x(\beta), s(\beta)) = (-\tilde{H}_x u \nabla x \partial_s u, \det H_x(u)\beta), \]

Such a comparison is difficult, regarding the fact that scale in a Poisson scale space has physical dimension length, whereas scale in the Gaussian scale space has physical dimension length squared.

As is shown in [1], there do not exist interior extrema (with respect to scale and position) in \( \alpha \)-scale spaces.
where \( \frac{1}{\det(H_xu)} \tilde{H}_xu = (H_xu)^{-1} \). This directly follows by application of the chain rule:

\[
\partial_\beta(\nabla_xu(x(\beta), s(\beta))) = H_xu(x(\beta), s(\beta))x(\beta) + \nabla_xu_s(x(\beta), s(\beta)) \frac{ds}{d\beta} = 0.
\]

The curvature at a top-point (or catastrophe point) \((x^*, s^*)\) along a critical curve is given by

\[
\kappa(x^*, s^*) = \frac{1}{\|w(x^*, s^*)\|^2 \det M(x^*, s^*)},
\]

where

\[
M(x^*, s^*) = \left. \begin{pmatrix} H_xu & \partial_s(\nabla_xu)^T \\ \nabla_x \det H_xu & \partial_s \det H_xu \end{pmatrix} \right|_{(x,s)=(x^*, s^*)}.
\]

In case the curvature at a catastrophe point is negative, the catastrophe is an annihilation and if the curvature is positive, the catastrophe is a creation. For proof and definition of \(w(x^*, s^*)\) we refer to Florack et al.\[9\]. To this end we notice that Florack’s derivation (only done for the case \(\alpha = 1\)) is straightforwardly generalized to the general case \(\alpha \in (0, 1]\). The critical curves through \(\alpha\)-scale spaces continuously depend on \(\alpha\), if scale is re-parameterized by \(s \mapsto C^* s^{1/(2\alpha)}\). Here the dimensionless constant \(C > 0\) is still arbitrary and in our evaluations it is chosen to scale the last annihilation at a fixed length. Furthermore we notice that the physical dimension of the re-parameterized scale equals length for all \(\alpha \in (0, 1]\). This continuous dependence is not surprising as the solutions (both on a bounded and unbounded domain) continuously depend on \(\alpha\), recall \(2\) and \(6\). However, in practice, it quite often happens that different extrema and saddles annihilate in \(\alpha\)-scale spaces, because of bifurcations with respect to the deformation of the scale space parameterized by \(\alpha\). Top-points can be created and annihilated with increasing \(\alpha\). Although, not considered here, this may be a point for further investigation from a Morse-theoretical point of view, see figure \(8\).
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Fig. 3. The critical curves through \( \alpha \) scale spaces of some arbitrary 2D-grey-value image only plotted within a box \([s_1, s_2] \times [x_1, x_2]\), \( s_1 < s_2, x_1 < x_2 \). From top left to bottom right, \( \alpha = 1 \) (Gaussian scale space) down to \( \alpha = 1/2 \) Poisson scale space, \( \alpha = 0.1, 0.95, 0.9, \ldots, 0.5 \). Notice that in the case \( \alpha = 1 \) there are two creations (two critical curves are bending below). In one of these cases (the closed loop) this creation is immediately followed by an annihilation of the same extremum and saddle. The critical curves continuously depend on \( \alpha \), but still for example at \( \alpha \approx 0.75 \) a (besides the bifurcation where the closed creation-annihilation-loop disappears) bifurcation arises where a scale space creation and annihilation meet each other. The last figure in the bottom row clearly illustrates the large difference in topology (due to the bifurcations in \( \alpha \)) within the Gaussian and Poisson scale space in a larger box within the same scale space.

3 Local Morse Theory for Gaussian Scale Space and Poisson Scale Space

In this subsection we give a short summary of the local Morse theory developed initially by James Damon,\[10\], for the diffusion equation and investigate how they translate to the case of the Poisson equation. It turns out that the generic topological changes in Poisson scale space correspond to the generic topological changes in Gaussian scale space. However, a fundamental difference between the frameworks, is that a Poisson scale space allows creations in scale spaces of 1D
signals \((d=1)\), whereas in the Gaussian scale space the diffusion equation does not allow creations in scale space of a 1D signal. Although these creations in a 1D Poisson scale space are stable/generic in the mathematical sense, they do not seem to occur frequently in practical situations\(^5\). Nevertheless, we will show both analytic and representative numerical examples of 1D-Poisson scale spaces including creation events.

Damon\(^10\) introduces the groups \(G = \mathcal{H}\), with \(\mathcal{H}\) the group of pairs \((\phi, c)\), with \(\phi: \mathbb{R}^{d+1} \to \mathbb{R}\) and \(c: \mathbb{R} \to \mathbb{R}\) diffeomorphisms, acting on the space of solutions of smooth functions \(S_\alpha\) satisfying the evolution equation \(\partial u / \partial s = -(-\Delta)^\alpha u\), \(\alpha \in (0, 1]\), (in particular \(\alpha\)-scale spaces) the group action

\[
g \cdot u(x, s) = u(\phi(x, s), \phi_2(s)) + c(s),
\]

where \(g \in \mathcal{H}\), the group of pairs \((x, s) \mapsto (\phi_1(x, s), \phi_2(s))\), where \(\phi_1: \mathbb{R}^{d+1} \to \mathbb{R}^d\) and \(\phi_2: \mathbb{R}^+ \to \mathbb{R}^+\) are diffeomorphisms and where \(\phi_2'(0) > 0\). Furthermore, Damon\(^10\) introduces the group \(G = \mathcal{IS}\), with \(\mathcal{IS}\) the group of pairs \((\phi, \psi)\), with \(\phi : \mathbb{R}^{d+1} \to \mathbb{R}\), \(\psi : \mathbb{R}^2 \to \mathbb{R}\) diffeomorphisms of the forms \(\phi(x, s) = (\phi_1(x, s), \phi_2(s))\) and \(\psi(y, t) = (\psi_1(y, t), t)\) with \(\phi_2'(0) > 0\) and \(\frac{\partial \psi}{\partial y}(0, 0) > 0\) and \(\psi_0(t) = 0\), acting on \(S_\alpha\) by

\[
g \cdot u(x, s) = \psi_1(u \circ \phi(x, s), s) + c = \psi_1(u(\phi_1(x, s), \phi_2(s)), s).
\]

By introducing these groups, he defines equivalence relations by means of

\[
u \sim v \text{ iff there exists a } g \in G \text{ such that } u = g \cdot v,
\]

i.e. two elements within \(S_\alpha\) are equivalent iff they lie on the same orbit, which yields \(\mathcal{H}\) and \(\mathcal{IS}\)-equivalence depending on the choice of group (action)\(^6\). However, here we follow an approach first formulated by J.C.van der Meer to singularity theory for the diffusion equation, which is similar to the case \(G = \mathcal{IS}\) similar cf. \([11]\). In this approach the action of the group \(G = \text{Diff}_d \times \text{Diff}_n\), with \(\text{Diff}_n\) the group of diffeomorphisms from \(\mathbb{R}^d\) to itself, acting on images \(f: \mathbb{R}^d \to \mathbb{R}\) is given by

\[
(g \cdot f)(x) = \psi \cdot f \cdot \phi^{-1}(x), \quad g = (\phi, \psi).
\]

Notice that this group rather acts on the set of images, rather than on the space of double sided \((\tilde{u}_f = e^{-(\sqrt{-1})s}f, s \in \mathbb{R})\) scale spaces, which are to be considered as 1-parameter deformations on images.

**Definition 1.** Let \(f: \mathbb{R}^d \to \mathbb{R}\). A 1-parameter deformation (or unfolding)\(^7\) of \(f\) is a continuous map \(u: \mathbb{R}^d \times \mathbb{R}^1 \to \mathbb{R}\) such that \(u(x, 0) = f(x)\).

---

5. They do seem to occur more frequently in the conjugate Poisson scale space.

6. Notice that with respect to \(\mathcal{IS}\)-equivalence, that the \(\mathcal{IS}\)-group action is rather similar to the \(\mathcal{H}\)-group, but the intensity may change over scale as well (by \(\psi_1\)) and the role of \(s > 0\) is no longer distinguished, so that by the equivalence relation one keeps track of local changes of an iso-intensity surface as it undergoes a transition and the intensity level of that critical point.

7. With unfolding one usually means the map \((x, s) \mapsto (u(x, s), s)\).
By considering 1-parameter deformations of the identity of the group $G$ we obtain the group $G_{un}$, which acts on 1-parameter deformations of images (in particularly scale spaces) by means of

$$(g \cdot u)(x, s) = \tilde{\psi}(u(\tilde{\phi}^{-1}(x, s)), s),$$

where $\tilde{\psi}(y, 0) = \psi(y)$ and $\tilde{\phi}^{-1}(x, 0) = (\phi^{-1}(x), 0)$. Notice that $(\tilde{g} \cdot \tilde{u}_f)(x, 0) = (g \cdot f)(x)$, for all $\tilde{g} \in G_{un}$ corresponding to a certain $g \in G$ for every $f \in L^2(\mathbb{R}^2)$.

**Definition 2.** Two deformations $u, v$ are called $\tilde{G}$-equivalent if $u$ lies within the orbit of $v$, or more precisely, there exists a $\tilde{g} \in \tilde{G}$ such that $u \sim v \iff u = \tilde{g} \cdot v$.

**Definition 3.** A function $f : \mathbb{R}^d \to \mathbb{R}$ is $G$-stable if any deformation is $G$-equivalent to the constant deformation. A deformation $u$ is $\tilde{G}$-stable\(^8\) if any deformation of $u$ is $G$ equivalent to the constant deformation.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is $G$-stable (or generic) iff the tangent space at $f$, $T_G(f)$ equals $E_0$, the space of all smooth germs of functions at zero.

**Lemma 1.** Thereby a double-sided scale space representation $\tilde{u}_f$ is $\tilde{G}$-stable iff either $f$ is stable or the co-dimension of $f$ in $E_0$ equals 1, $\dim(E_0/T_G(f)) = 1$, and $\frac{\partial}{\partial s} \tilde{u}_f |_{\tau=0}$ generates the complement.

However, we are merely interested in single sided scale spaces where scale is a strictly positive parameter, $s > 0$, as we do not want to include (ill-posed) de-blurring.

This means that in the co-dimension one case we can only generate half\(^9\) of the remainder in the tangent space.

This is the reason for a distinction between one-sided and two sided stability. In the case of two sided stability the function (or rather germ) $f$ was already stable, whereas in the second case the co-dimension of $f$ in $E_0$ equals 1. For further details, see Van der Meer\(^{[11]}\).

### 3.1 A Partition of Equivalence Classes of $\tilde{G}$-Stable Gaussian Deformations

The space of $\tilde{G}$-stable Gaussian Deformations $S_{\alpha=1}$ can be partitioned into equivalence classes due to the equivalence relation given by (2). First we consider the case $d \geq 2$. Then the equivalence classes (germs) are represented by one of the following functions:

1. Two sided stable germs:
   - (a) $u(x, s) = x_1$, (submersion)

\(^8\) The notion of stability can be rephrased as follows: A function $u$ is stable if all functions that are close to $u$ (in an appropriate topology) are $\tilde{G}$-equivalent to $u$.

\(^9\) The sign of $\frac{\partial u}{\partial s}|_{s=0}$ relative to becomes relevant in the definition of $G$-equivalence.
(b) \(2ds + \sum_{i=1}^{d} x_i^2\)

(c) \(\sum_{i=1}^{d} a_i x_i^2\), with \(\sum_{i=1}^{d} a_i = 0\) all \(a_i \neq 0\). (classified by the signs of \(a_i\))

2. One sided stable germs:

(a) \(x_1^3 - 6x_1(x_2^2 + s) + Q(x_2, \ldots, x_d, s)\), (creations of critical points),

(b) \(x_1^3 + 6s x_1 + Q(x_2, \ldots, x_d, s)\), (annihilations of critical points),

with \(Q(x, s) = \sum_{k=2}^{d} \epsilon_k(x_k^2 + 2s)\), with \(\epsilon_k = \pm 1\) for \(k = 2, \ldots, n\).

The two-sided stable germs are of less interest since in all of these cases \(T_G(f) = \mathcal{E}_0\) and thereby \(T_G(f) + \mathbb{R}^+ \frac{\partial u}{\partial s} \big|_{s=0} = T_G(f) = \mathcal{E}_0\). So here we have no bifurcations of the critical paths in scale space.

It follows from Lemma 4 that the only bifurcations the image \(f\) can undergo as a consequence of Gaussian blurring are given by singularities of co-dimension 1. The standard form of a co-dimension one function is given by

\[C_1(x) = x_1^3 + \sum_{i=2}^{d} a_i x_i^2, a_i \neq 0,\] (9)

with universal deformation \(x_1^3 + tx_1 + \sum_{i=2}^{d} a_i x_i^2\). For \(t > 0\) there are no critical points while for \(t < 0\) there are two critical points, a saddle and an extremum. This is known as the cusp catastrophe [12].

The Gaussian deformation of \(C_1\) is given by \((G_s * C_1)(x) = x_1^3 + 6s x_1 - 2(\sum_{i=2}^{d} a_i)s + \sum_{i=2}^{d} a_i x_i^2\). This yields the germs describing the annihilation of critical points in a Gaussian scale space. Notice that the complement to the tangent space is spanned by the vector \(\Delta C_1(x) = 6 x_1 \equiv x_1\).

Consider \(C_2(x) = x_1^3 - 6x_2^2 x_1 + \sum_{i=2}^{d} a_i x_i^2\). Although \(C_2\) is equivalent to \(C_1\) their Gaussian deformations are not one-sided equivalent: The Gaussian deformation of \(C_2\) equals \(G_s * C_2(x) = x_1^3 - 6x_2^2 x_1 - 6x_1 s - 2(\sum_{i=2}^{d} a_i)s + \sum_{i=2}^{d} a_i x_i^2\), so now the complement to the tangent space is spanned by the vector \(\Delta C_2(x) = -6 x_1 \equiv -x_1\).

The case \(d = 1\) can be treated in an analogue matter, except for the annihilation germ, where no such \(x_2\) is at hand. Creations can not occur in a Gaussian scale space. Recall, to this end that that the curvature at a catastrophe point \((x\ast, s\ast)\) \((u_{xx}\ and \ u_x\ vanish)\) equals \(\kappa = \frac{1}{\sqrt{\det M}}\) \(\det M\), see (7). In 1D the matrix \(M\) is given by

\[M(x\ast, s\ast) = \begin{pmatrix} u_{xx}(x\ast, s\ast) & u_{xs}(x\ast, s\ast) \\ u_{sx}(x\ast, s\ast) & u_{ss}(x\ast, s\ast) \end{pmatrix},\]

and since \(u_{xx}(x\ast, s\ast) = 0\) it directly follows from

\[\det M(x\ast, s\ast) = -u_{xs}(x\ast, s\ast)u_{xx}(x\ast, s\ast) = -(u_{xx}(x\ast, s\ast))^2\]
that the curvature is always negative, allowing only annihilations and no cre-
ations. Notice that this argument does not hold in a Poisson scale space, where
\[
\det M(x^*, s^*) = u_{xx}(x^*, s^*)u_{xxx}(x^*, s^*) = -\frac{1}{2} \left( \frac{d}{ds} u_{xx} \right)^2(x^*, s^*) = -\frac{1}{2} \left( \frac{d}{ds} u_{xxx} \right)^2(x^*, s^*),
\]
where \( v \) denotes the conjugate Poisson scale space, is not a priori negative.

3.2 A Partition of \( G \)-Stable Poisson Deformations

Following the same line as for the Gaussian case we obtain the following partition
of \( \tilde{G} \)-stable Poisson Deformations:

1. Two sided stable germs:
   (a) \( u(x, s) = x_1, \) (submersion)
   (b) \( -2ds^2 + \sum_{i=1}^{d} x_i^2 \)
   (c) \( \sum_{i=1}^{d} a_i x_i^2, \) with \( \sum_{i=1}^{d} a_i = 0 \) all \( a_i \neq 0. \) (classified by the signs of \( a_i \))

2. One sided stable germs:
   (a) \( x_3^1 - 3s^2 x_1 - s x_1 + Q(x_2, \ldots, x_d, s), \) (creations of critical points),
   (b) \( x_3^1 - 3s^2 x_1 + s x_1 + Q(x_2, \ldots, x_d, s), \) (annihilations of critical points),
   with \( Q(x, s) = \sum_{k=2}^{d} \epsilon_k (x_k^2 - s^2), \) where \( \epsilon_k = \pm 1 \) for \( k = 2, \ldots, d. \)

The annihilation and creation germ are obtained by means of harmonic extension
of the cusp catastrophe \( C_1 \) given by (9). To this end we notice that \( \Re((x_1^3 \pm is)^3) = x_1^3 - 3s^2 x_1. \) At this point it should be noticed that the Laplace operator factorizes in 2 different ways
\[
\Delta_{d+1} = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial x_1^2} + \Delta_{d-1} = \left( \frac{\partial}{\partial s} - \sqrt{-\Delta_d} \right) \left( \frac{\partial}{\partial s} + \sqrt{-\Delta_d} \right) = \left( \frac{\partial}{\partial s} + i \frac{\partial}{\partial s} \right) \left( \frac{\partial}{\partial s} - i \frac{\partial}{\partial s} \right) + \Delta_{d-1},
\]
As we already noticed, the first factorization tells us that upward harmonic
extension of an image \( f \in L_2(\mathbb{R}^d) \) under the additional requirement that the
harmonic extension should uniformly vanish as \( s \to \infty \) is equivalent to solving
the first order pseudo differential evolution system (1) for \( \alpha = 1/2, \) leading to
Poisson scale space. Here we only consider local behavior and do not have the
additional requirement at hand. Therefore harmonic extension of the 1 cusp
catastrophe (with co-dimension 1) is not sufficient. For example, a creation by
harmonic extension could be due to a Poisson de-blurring. More precisely, the
Poisson deformation could in principle be obtained by means of the evolution
generated by \( +\sqrt{-\Delta} \) rather than \( -\sqrt{-\Delta}. \) At this point, in comparison to the
Gaussian case, there arises a technical problem as the convolution of the cusp
catastrophe with the Poisson kernel does not exist.

10 Annihilations become creations if \( s \mapsto -s. \)
Therefore (as we are interested in local behavior only) we compute $(H^d_\alpha * g^d_{b,\varepsilon})(x)$, where $H^d_\alpha$ denotes the $d$ dimensional Poisson/Cauchy kernel and

$$g^d_{b,\varepsilon}(x) = g^1_{b,\varepsilon}(x_1) = (x_1^3 + \varepsilon x_1)1_{[-b,b]}(x_1), x = (x_1, \ldots, x_d),$$

where $\varepsilon = \pm 1$ and $1_{[-b,b]}(x_1) = 1$ if $|x_1| < b$ and 0 elsewhere. Some computation yields

$$(H^d_\alpha * g^d_{b,\varepsilon})(x) = (H^1_\alpha * g^1_{b,\varepsilon})(x_1)$$

$$= \frac{1}{2\pi} \{8 x_1 b s - 2(x_1^3 - 3x_1 s^2 + \varepsilon x_1) \left[ \arctan \left( \frac{x_1-b}{s} \right) - \arctan \left( \frac{x_1+b}{s} \right) \right] - s(s^2 - 3x_1^2 + 2) \log \left( \frac{(x_1-b)^2 + s^2}{(x_1+b)^2 + s^2} \right) \}.$$

If we now omit the order $O(b)$-term and take the limit $b \to \infty$ we indeed obtain the annihilation ($\varepsilon = 1$) and creation ($\varepsilon = -1$) germs $x_1^3 - 3x_1 s^2 + \varepsilon x_1$, $\varepsilon = \pm 1$.

Further we notice that in contrast to the Gaussian case creations are $\mathcal{G}$-stable in the 1D-case $d = 1$. Notice that in the 1D-creation case the critical paths of the above germ are given by $x(s) = \pm \sqrt{s^2 + \frac{1}{3} s}$, $s > 0$, whereas the critical paths of the annihilation germ are given by $x(s) = \pm \sqrt{s^2 - \frac{1}{3} s}$, $s < 0$.

For a simple analytic example of a 1D-creation (at scale $s = 0$) in Poisson scale space, where the initial condition only consists of 3-delta spikes, see figure 4. To this end we notice that in stead of $\delta$-spikes (which are distributions, not functions) one may as well take the step function $f = -1_{(-d,d)} + 1_{(a-d,a+d)} + C * 1_{(-b,-b+d)}$, with $a, b > 2 * d$ as initial condition. In this case the Poisson scale space can also be computed analytically and the scale space is given by

$$u^0_f(x,s) = \frac{1}{\pi} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \left( \frac{1}{s} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right),$$

which also has five extrema at $s = 1/4$, for example at $\approx -1.50, 0.128, 0.304, 2.96$ and 4.00, if $a = 4, b = 1.5, d = 0.1, C = 25$. For a numerical example of a 1D-creation within (at a scale $s > 0$) Poisson scale space see figure 4.

4 Causality in Gaussian and Poisson Scale Space

Another difference the deeps structure between Gaussian scale space and other $\alpha$-scale spaces ($\alpha \neq 1$) is causality. There exists two types of causality, see Definition 5, the Gaussian scale space is the only $\alpha$-scale space which satisfies strong causality. To this end we refer to figure 7 and we note that it is already shown by Hummel[13] that strong causality is equivalent to the maximum principle, see Definition 5, which is well-known to hold for the diffusion system. The maximum principle follows directly follows by the Koenderink’s principle that states that $u_s(x,s) \Delta u(x,s) > 0$ at spatial extrema, which only holds in a Gaussian scale space, and which guarantees that local extrema do not enhance.

\[11\] In [II] we have shown that all $\alpha$-scale spaces satisfy weak causality.
Fig. 4. Top Row: From left to right, slices of the Poisson scale space, $u_f^\alpha(\cdot, s)$, of a signal $f$ consisting of only 3 δ-spikes, with analytic solution $u_s^{\alpha=1/2}(x) = K_s^{\alpha=1/2}(x-a) + K_s^{\alpha=1/2}(x+b)$, $a = 8, b = 2, c = 10$ at $s = 0.001, s = 0.05, s = 0.1, s = 0.15$. Bottom Row: Corresponding Gaussian scale space $u_s^{\alpha=1}(x) = K_s^{\alpha=1}(x-a) + K_s^{\alpha=1}(x+b)$, $s = (1/2)\sigma^2$ at $\sigma = 0.2, \sigma = 0.7, \sigma = 0.12, \sigma = 0.17$. This simple analytical example gives a clear illustration of a creation (3 singular points at $s = 0$ become 5 singular points at $s > 0$) in Poisson scale space at $s = 0$, whereas in the Gaussian scale space the number of extrema remains 3 as it should as creations cannot occur in a 1D-Gaussian scale space. Notice that the extremum at $x \approx 1.5$ increases in value in the Poisson scale space, due to the Koenderink principle, which is related to the difference in causality (and maximum principle) between Gaussian and Poisson scale space.

Fig. 5. Top down, left to right, slices of a numerical implementation of a 1D-Poisson scale space, $u_f^\alpha(\cdot, s)$, of a numerical signal $f$, with $s = 0, 0.4, 0.8, 1.2, 1.6, 2.0$. It is clearly seen that a creation event takes place at $(x, s) \approx (500, 0.5)$.

**Definition 4.** Weak Causality: Any scale space isophote $u(x, s) = \lambda$ is connected to the ground plane, i.e. it is connected to a point $u(x, 0) = \lambda$.

Strong Causality Constraint: For every $s_1 \geq 0$ and $s_2 > 0$ with $s_2 > s_1$ the intersection of any connected component of an isophote within the domain $\{(x, s) \in \mathbb{R}^d \times \mathbb{R}^+ | x \in \mathbb{R}^d, s_1 \leq s < s_2\}$ with the plane $s = s_1$ should not be empty.
Definition 5. (*Cylinder Maximum Principle.*) Let \( \Omega \) be a (arbitrary) bounded subset of \( \mathbb{R}^d \) and \( s_1 > 0 \) such that \( u \) is continuous on \( \overline{\Omega} \times [0, s_1] \), then \( u \) attains its maximum or minimum in say \( (x, s) \in \overline{\Omega} \times [0, s_1] \). Either we must have \( s = 0 \) or \( x \in \partial \Omega \).

Fig. 7. Isophotes of various scale space representations of a signal consisting of 1 small delta spike between two larger delta spikes. Top row: \( \alpha = 0.5 \) (Poisson scale space), \( \alpha = 0.6, \alpha = 0.7 \), bottom row: \( \alpha = 0.8, \alpha = 0.9 \) and \( \alpha = 1 \) (Gaussian scale space). The \( \alpha \) scale spaces are sampled according to \( s_\alpha = e^{\alpha \tau_n} \), with equidistant \( \tau_n \). To this end we notice that both \( (s_\alpha)_{\alpha}^{1/2} = \sigma \) and \( \sqrt{s_1} = \sigma \) have dimension [Length]. The stretching of the isophotes as \( \alpha \) increases is of no importance. The above figure shows that for each \( \alpha \in (0, 1) \) there exist locally concave critical isophotes.

5 Conclusion

There exists a simple and strong connection between \( \alpha \)-scale spaces and the corresponding vector scale spaces (and their flow fields). In particular between the case of Poisson scale space \( (\alpha = 1/2) \), where this vector scale space is the well-known Monogenic scale space, and the case of Gaussian scale space \( (\alpha = 1) \), where this vector scale space extension equals the first order jet, which is somewhat surprising concerning the different physics involved (respectively potential
physics and heat physics). This raises the question whether these \(\alpha\)-scale spaces are essentially different for multi-scale image analysis, as methods in one framework are easily translated to methods in the other frameworks. We approached this question only from a topological point of view and conclude that the deep structure of \(\alpha\)-scale spaces, although continuously deformed by \(\alpha\), provided scale is properly re-parameterized, is rather different:

- Experiments on daily life images often show that different extrema and saddles annihilate in \(\alpha\)-scale spaces. Moreover, creation events that take place in one scale space need not take place in the other.
- By applying Morse theory on Poisson scale space and Gaussian scale space we deduce that creations are generic events in 1D-Poisson scale space (illustrated by both analytic and numerical examples), whereas they can not occur in Gaussian scale space.
- Isophotes through critical points do behave differently in Gaussian scale space than in the other \(\alpha\)-scale spaces, where they behave similarly.

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