Chapter 1

Modeling: an introduction

Taken from lecture notes 8E010 ”Models of physical processes” by P.P.J. van den Bosch.

1.1 Models

Models are used to understand and predict the behavior of real-life processes. As soon as we can describe these real-life processes sufficiently accurate with the aid of a mathematical model, analyzing and understanding these processes becomes easier. When the model is sufficiently accurate, it is possible to make predictions on future behavior of the process (e.g. weather forecasting). As such, models can be considered as our knowledge of real-life processes.

One of the most difficult steps in modeling is to determine the system boundary in the real-life process with inside the components and functions to be modeled and outside other parts that have to be neglected. If the inside part is too small, a simple model results, but its value to describe or predict the behavior of the process is too restricted or too inaccurate. If the inside part is too large, a very complex model can result which is too complicated to deal with. Both situations are not attractive and have to be avoided. Thus, the model should contain only those components or functions that are really important for our study. A consequence will be that there are many different models for the same process, each model for its own application. Moreover, any model will always be smaller and describe less details than the real-life process, i.e.,

$$\text{Model} \subset \text{Process}$$

Another important factor deals with the time range. If we want to study physiological phenomena, the time scale of the processes can be very fast (milliseconds for electrical activities) to a billion years (dealing with evolution), so from 10^{-3} to 10^{16} seconds, a range of 10^{19} between smallest and largest time interval. This is too large! Reasonable models have a range of maximal 10^4. A larger range will introduce complexity, large calculation times, and results that cannot be compared. In general, processes that are faster than the selected time range of the model are assumed to have reached an equilibrium, i.e., they are assumed to be constant. Processes that are slower than the selected time range of the model are also considered as constant in the time considered by the model.

Example
In studying the changes of the lung during evolution, the time range has to be at least 100, 000 years or more. Then, also describing the periodic breathing of 10 seconds in the same model is both unrealistic and technically almost impossible.

In general we can distinguish between two different kinds of models, namely

1. *Structural* or *mechanistic* models that have a close resemblance with the underlying process, for example, electric circuits, drawings of hydraulic or mechanical systems. Real physical components and their interconnections can be distinguished. The internal variables are clearly visible. These models incorporate all available knowledge. They are quite complex with many, sometimes unknown, parameters.

2. *Functional* models that just reflect the external behavior of a process among its inputs and outputs. The internal construction or operation is of less importance. Main goal is to represent its external behavior. The internal processes can be hidden. Examples of functional models are mathematical models such as differential equations, state space models, transfer functions, impulse responses or their graphical representations as block diagram. Functional models are more abstract, and, consequently, more generally applicable.

Mathematical models can be described by equations. For our purpose, we distinguish between *algebraic* equations, *differential* equations, and *state space* equations:

In *algebraic equations*, the relation between the variables is instantaneous, e.g.,

- The relation between force f and displacement x of a spring is described by $f = k \cdot x$, for spring constant k.
- The relation between pressure difference Δp and flow ϕ in a pipe is described by $\phi = c \cdot \sqrt{\Delta p}$, for some constant c.

In *differential equations*, the time behavior, or *dynamics*, of variables is important. For example, when applying an input to a dynamic process, it takes some time before the effect becomes visible and reaches its maximum effect. Sometimes, it disappears again. Such processes are called *dynamic*.

Examples:

- Drinking one or many glasses of beer results in a gradually increase of alcohol in the blood and after some hours in a hang-over. Many hours later all effects will disappear.
- A charged capacity can be discharged by a resistor, but it takes some time before voltage has dropped below a certain value.
- Pressing the gas pedal in a car will result in an increase of velocity, but not immediately the final speed will be reached.

Differential equations are appropriate tools to describe dynamical phenomena. An example of a differential equation is

$$\dddot{y}(t) + 2\ddot{y}(t) + 5\dot{y}(t) + 10y(t) = 4\dot{u}(t) + 8u(t) \tag{1.1}$$

In general, we call the input variable $u(t)$ and the output variable $y(t)$.

State space equations are also used to describe the time behavior or dynamics of processes. Differential equations can be rewritten as state space equations and vice versa. For example, the previous differential equation yields the following state space model. In addition to the input $u(t)$ and the output $y(t)$, the internal variable $x(t)$, the *state*, is introduced. An n-th order differential equation will yield a state space model with an n-dimensional state vector $x(t)$ and n first-order differential equations, one for each of the n elements of $x(t)$. Moreover, an algebraic output equation defines the output $y(t)$ as a function of the state $x(t)$ and input $u(t)$, i,e.,

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t))\\ y(t) &= g(x(t), u(t))\end{aligned}$$

For linear functions f and g, the state equations can be formulated with vectors and matrices, for example:

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{pmatrix} = \begin{pmatrix}-2 & -5 & -10\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix} + \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\cdot u(t) \tag{1.2}$$

$$y(t) = \begin{pmatrix}0 & 4 & 8\end{pmatrix}\cdot\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix} + \begin{pmatrix}0\end{pmatrix}\cdot u(t) \tag{1.3}$$

This state equation expresses the same dynamical behavior between $u(t)$ and $y(t)$ as expressed by the differential equation (1.1).

1.1.1 Differential equations

In differential equations, not only the value of a variable $y(t)$ is important, but also its rate of change, which is expressed as

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \dot{y}(t) = \lim_{\Delta t\to 0}\frac{y(t+\Delta t) - y(t)}{\Delta t} \tag{1.4}$$

This rate of change of $y(t)$ is the *time derivative* of $y(t)$, i.e., the slope of $y(t)$ when it is drawn as a function of time t. In addition to the first-order derivative $\dot{y}(t)$, also higher-order derivatives of $y(t)$ can occur in a differential equation, such as the second-order $\ddot{y}(t)$, third-order derivative $\dddot{y}(t)$ or n-th order derivative $y^{(n)}(t)$.

The *order n* of a differential equation is determined by the highest derivative of $y(t)$ in the equation. For calculating a unique solution of an n-th order differential equation, the values of n *initial conditions* have to be known, i.e., $y(0)$, $\dot{y}(0)$, $\ddot{y}(0)$, $\ldots$, $y^{(n)}(0)$.

Differential equations turn out to be valuable mathematical descriptions for studying, analyzing and synthesizing dynamical processes.

Examples

First-order differential equations with 1 initial condition and their solution:

Differential equation	**Initial condition**	**Solution** $y(t)$**,** $t \geq 0$
$\dot{y}(t) = 0$	$y(0) = y_0$	$y(t) = y_0$
$\dot{y}(t) = a$	$y(0) = y_0$	$y(t) = y_0 + at$
$\dot{y}(t) + ay(t) = 0$	$y(0) = y_0$	$y(t) = y_0 \cdot \exp(-at)$
$\dot{y}(t) + ay(t) = b$	$y(0) = y_0$	$y(t) = (b/a)(1 - \exp(-at)) + y_0 \cdot \exp(-at)$

Second-order differential equations with 2 initial conditions and their solution:

Differential equation	**Initial condition**	**Solution** $y(t)$**,** $t \geq 0$
$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 1$	$y(0) = \dot{y}(0) = 0$	$y(t) = 1 - 2\exp(-t) + \exp(-2t)$

1.1.2 State space equations

In dynamic models, we can define a *state vector*, besides the input and output. The state is an internal variable of the model, and represents, in general, the physical buffers of a process. If we know the state $x(t)$ at time $t = 0$, i.e., $x(0)$, and the value of input $u(t)$ for $t = 0$, then both $x(t)$ and $y(t)$ for $t = 0$ are fully defined by the state (space) equations

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

These state equations define the first time derivative $\dot{x}(t)$ of the state $x(t)$ as an algebraic function of ONLY the state $x(t)$ itself and the input $u(t)$. The output $y(t)$ is defined as an algebraic function of ONLY the state $x(t)$ and the input $u(t)$. When the model is of n-th order, the state $x(t)$ will be a vector with n elements. When the model has one input and one output, both $u(t)$ and $y(t)$ are scalar functions. For obtaining a unique solution $x(t)$, and thus $y(t)$, there has to be an initial condition $x(0)$ for $t = 0$. Vector $x(0)$ has n elements.

When both functions $f(x, u)$ and $g(x, u)$ are linear in both x and u, linear state equations result:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

1.1.3 Vectors and matrices

State space models utilize vectors and matrices. These are nice and attractive components from linear algebra to simplify variables and equations. Vectors have n elements, either ordered as a row

$(x_1, x_2, x_3, \ldots, x_n)$ or as a column $(x_1, x_2, x_3, \ldots, x_n)^\mathsf{T}$. A matrix has, for example, n rows and m columns. The dimension of a row vector is $(1 * m)$, a column vector has dimension $(n * 1)$, and the matrix has dimension $(n * m)$.

Example
The matrix equation $y = Ax$ represents the following equations when y is a 2-dimensional and x a 3-dimensional column vector:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The associated equations are:

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{aligned}$$

Clearly, $y = Ax$ is a much shorter notation. Some more matrix characteristics: $A \cdot A^{-1} = I$, with I the unity matrix. When $A = B^\mathsf{T}$ (T from transpose), the following is valid: $a_{kl} = b_{lk}$.

1.2 Deriving a model

1.2.1 Analogies among different physical fields

In several physical fields the same type of variables and components can be recognized. These variables always occur as pairs. They are selected such that their product yields power [W]. The interaction among physical components can best be understood by considering the exchange of power among them. Understanding one field makes understanding the other fields more easy. Table 1.1 describes the analogies among the fields electrical, mechanical, hydraulic, acoustic and thermal systems. The first variable is called *effort* or across variable (voltage, force, torque, pressure, temperature). The second one is the *flow* or through variable (current, velocity or flow). The product of *effort* and *flow* yields the physical quantity *power P* [W].

In these five physical fields, or application areas, the following generic components can be distinguished with a unique relation between their *effort* (e) and *flow* (f) variables:

Resistance R	$e = f \cdot R$
Capacity C	$\dot{e} = \frac{1}{C} \cdot f$ or $e(t) = e(0) + \frac{1}{C} \int_{\tau=0}^{t} f(\tau)\mathrm{d}\tau$
Inductance L	$\dot{f} = \frac{1}{L} \cdot e$ or $f(t) = f(0) + \frac{1}{L} \int_{\tau=0}^{t} e(\tau)\mathrm{d}\tau$

For electrical systems the effort is the voltage u [V] and the flow variable is the current i [A]. Then these 3 generic equations become:

Resistance R [Ω]	$u = i \cdot R$
Capacity C [F]	$\dot{u} = \frac{1}{C} \cdot i$ or $u(t) = u(0) + \frac{1}{C} \int_{\tau=0}^{t} i(\tau) d\tau$
Inductance L [H]	$\dot{i} = \frac{1}{L} \cdot u$ or $i(t) = i(0) + \frac{1}{L} \int_{\tau=0}^{t} u(\tau) d\tau$

Only a resistance (R or b) dissipates power and so heats up. This thermal power can, in general, not be recovered and will leave the system. Capacity and inductance (C and L) are buffers, so they can store power [W] as energy [J] = [W s]. The amount of energy stored is elucidated in Table 1.1. Power enters and leaves a buffer without a loss. When making a model, each buffer introduces a state.

The number of states equals the number of buffers in a model.

1.2.2 SI units

When dealing with physical systems, it is a necessity to take care of appropriate units for all variables and parameters. Much confusion and even errors can be avoided if proper usage is made of these units. There is an international scientific standard of units (SI: systeme internationale). Unhappily, the USA utilizes other units (miles in stead of meters, pounds in stead of kg, etc.) which might give considerable confusion. The seven basic SI units are

Quantity	**Symbol**	**Unit**	**Symbol**
Length	l	meter	[m]
Mass	m	kilogram	[kg]
Time	t	second	[s]
Electric current	i	Ampere	[A]
Temperature	T	Kelvin	[K]
Light strength	I	candela	[cd]
Quantity matter	n	mole	[mol]

In Table 1.1, additional variables with their units are introduced and defined for specific application areas such electrical, mechanical, hydraulic and thermal variables. For each of these areas the two most important variables are:

Electrical	**Mechanical translation**	**Mechanical rotation**	**Hydraulic / acoustic**	**Thermal**
Voltage: u [V]	Force: f [N]	Torque: M or τ [Nm]	Pressure: p [N/m^2]	Temperature: T [K]
Current: i [A]	Velocity: v [m/s]	Angular speed: ω [rad/s]	Volume flow: ϕ [m^3/s]	Heat flow: q [W]

In addition to the components R, C and L, there are more complex components that take care of power conversion within or between two fields, for example transformers, motors and pumps. When these components are ideal, there is no power loss. Consequently, the power at one side has to be equal to the power at the other side. These components are illustrated in Table 1.2.

1.2.3 Deriving a model in 4 steps

1a:	Define input u	Inputs are external variables that cannot be influenced
1b:	Define state x	Each buffer is a state
2:	Define state equation $\dot{x}(t) = f'(x(t), u(t), z(t), \ldots)$	Each state variable has its own state equation (Table 1.1)
3:	Remove all variables in right hand side, except for u and x $\dot{x}(t) = f(x(t), u(t))$	Use additional equations for this removal Verify signs of state variables in equations
4:	Define output equation, only as function of u and x $y(t) = g(x(t), u(t))$	Outputs are given by the problem formulation or application

A state space model can always be found by using the following 4 steps:

1. *Define inputs u and determine the buffers.* Each buffer has a state variable x_i associated with it. Define this state. The number of independent states becomes the order of the state space equation.

2. Each state variable has its own *state equation* (see Table 1.1).

3. *Remove/eliminate all variables that are no inputs u and no states x* from the right hand side. This goal can always be achieved by using additional equations, for example applying Kirchhoff's laws, component equations, etc. Verify signs of state variables in right hand side.

4. *Define the output equation* as function of only the input and the state.

Rule for testing signs of right hand side:

The state variables at the right hand side that correspond with the state variable at the left side will ALWAYS have a minus sign. This rule is a characteristic of all physical (passive) processes.

Remark: When an input u is used that depends on the state variables, this rule does not apply. The sign of the state variable in the input is then uncertain.

1.2.4 Electrical processes

Each capacity C_i yields a state u_i

Each inductance L_i yields a state i_i

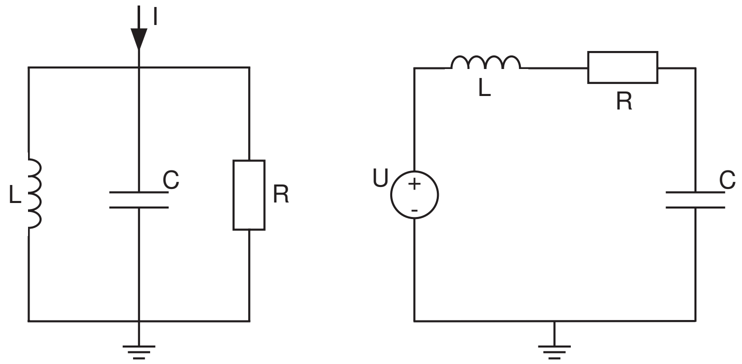

Figure 1.1: Left: A parallel connection of a resistor R, an inductance L and a capacitor C with a current source I. Right: A series connection of a voltage source U with a resistor R, an inductance L, and a capacitor C.

Each junction of wires: apply Kirchhoff's current law:

net current in any connection is zero or $\sum_{j=1}^{n} i_j = 0$

Each loop of components: apply Kirchhoff's voltage laws

net voltage along a loop in a circuit is zero or $\sum_{j=1}^{n} u_j = 0$

Examples
A parallel connection of a resistor R [Ω], an inductance L [H] and a capacitor C [F] with a current source I [A] (Figure 1.1, left), yields a model with input I and states u_C [V] and current i_L [A]:

$$
\begin{aligned}
\dot{u}_C &= \frac{1}{C}(I - i_L - \frac{u_C}{R}) \\
\dot{i}_L &= \frac{1}{L}u_C
\end{aligned}
$$

A series connection of a voltage source U [V] with a resistor R [Ω], an inductance L [H] and a capacitor C [F] (Figure 1.1, right), yields a model with input U and states u_C [V] and current i_L [A]:

$$
\begin{aligned}
\dot{u}_C &= \frac{1}{C}i_L \\
\dot{i}_L &= \frac{1}{L}(U - i_L R - u_C)
\end{aligned}
$$

Table 1.1: Analogies among different physical fields

	Electrical	**Mechanical translation**	**Mechanical rotation**	**Hydraulic / acoustic**	**Thermal**
Effort Flow	u [V] i [A]	f [N] v [m/s]	M or τ [Nm] ω [rad/s]	p [N/m^2] ϕ [m^3/s]	T [K] q [W]
Power P [W]	$P = u \cdot i$	$P = f \cdot v$	$P = \tau \cdot \omega$	$P = p \cdot \phi$	$P = q$!!
Resistance (damper) R	R [Ω] $u = i \cdot R$	b [Ns/m] $f = v \cdot b$	b [Nms/rad] $\tau = \omega \cdot b$	R_H [Ns/m^5] lin: $p = \phi \cdot R_H$ non-lin: $p = R'_H \cdot \phi^2$ $\phi = k\sqrt{p}$	R_T [K/W] $T = q \cdot R_T$
Capacity (spring) C	C [F] $\dot{u} = \frac{1}{C} \cdot i$ $(i = i_{\text{in}} - i_{\text{out}})$	k [N/m] $\dot{f} = k \cdot v$ $(f = k \cdot x)$	k [Nm/rad] $\dot{\tau} = k \cdot \omega$ $(\tau = k \cdot \theta)$	C_H [m^5/N] $\dot{p} = \frac{1}{C_H} \cdot \phi$ $(\phi = \phi_{\text{in}} - \phi_{\text{out}})$	C_T [J/K] $\dot{T} = \frac{1}{C_T} \cdot q$ $(q = q_{\text{in}} - q_{\text{out}})$
Inductance (mass, inertia, inertance) L	L [H] $\dot{i} = \frac{1}{L} \cdot u$	m [kg] $\dot{v} = \frac{1}{m} \cdot f$ $(f = m \cdot a)$	J [kg m^2] $\dot{\omega} = \frac{1}{J} \cdot \tau$ $(\tau = J \cdot \alpha)$	L_H [kg/m^4] $\dot{\phi} = \frac{1}{L_H} \cdot p$	
Buffer with **state variable**	$C \rightarrow \mathbf{u}$ $\dot{u}_C = \frac{1}{C} \cdot i$ $L \rightarrow \mathbf{i}$ $\dot{i}_L = \frac{1}{L}\Delta u$	$m \rightarrow \mathbf{v}, \mathbf{x}$ $\dot{x} = v$ $\dot{v} = \ddot{x} = \frac{1}{m}\sum f_i$	$J \rightarrow \omega, \theta$ $\dot{\theta} = \omega$ $\dot{\omega} = \ddot{\theta} = \frac{1}{J}\sum \tau_i$	$C_H \rightarrow \mathbf{p}$ $\dot{p} = \frac{1}{C_H}\sum \phi$ $L_H \rightarrow \phi$ $\dot{\phi} = \frac{1}{L_H} \cdot \Delta p$	$C_T \rightarrow \mathbf{T}$ $\dot{T} = \frac{1}{C_T}\sum q_i$
Energy stored in C-buffer [J]	$\frac{1}{2}C \cdot \mathbf{u^2}$	$\frac{1}{2}k \cdot \mathbf{x^2}$	$\frac{1}{2}k \cdot \theta^2$	$\frac{1}{2}C_H \cdot \mathbf{p^2}$	$C_T \cdot \mathbf{T}$
Energy stored in L-buffer [J]	$\frac{1}{2}L \cdot \mathbf{i^2}$	$\frac{1}{2}m \cdot \mathbf{v^2}$	$\frac{1}{2}J \cdot \omega^2$	$\frac{1}{2}L_H \cdot \phi^2$	

Table 1.2: Power conversion

On both sides the same variables (effort or flow):

E/E transformer	$u_2 = \alpha \cdot u_1,\ i_1 = \alpha \cdot i_2$	$P_1 = u_1 \cdot i_1 = u_2 \cdot i_2 = P_2$
Mr/Mr gear box	$\tau_2 = \alpha \cdot \tau_1,\ \omega_1 = \alpha \cdot \omega_2$	$P_1 = \tau_1 \cdot \omega_1 = \tau_2 \cdot \omega_2 = P_2$
Mt/H pump	$f = A(p_2 - p_1),\ \phi = A \cdot v$	$P_M = f \cdot v = (p_2 - p_1) \cdot \phi = P_H$

On both sides different variables:

E/Mr DC-motor	$\tau = \alpha \cdot i,\ u = \alpha \cdot \omega$	$P_E = u \cdot i = \alpha\omega \cdot \tau/\alpha = \omega \cdot \tau = P_M$
E/Mt linear motor/loudspeaker	$f = \alpha \cdot i,\ u = \alpha \cdot v$	$P_E = u \cdot i = \alpha v \cdot f/\alpha = v \cdot f = P_M$
Mr/H centrifugal pump	$p_2 - p_1 = \alpha \cdot \omega,\ \tau = \alpha \cdot \phi$	$P_H = (p_2 - p_1)\phi = \alpha\omega \cdot \tau/\alpha = \omega \cdot \tau = P_M$