Natural limits on observations

He who asks a question is a fool for five minutes;
he who does not ask a question remains a fool forever.
Chinese Proverb

Init

```
<< MathVisionTools`;
Off[GaussianDerivative::"scalefail"];
Unprotect[gD]; Clear[gD];
gD[im_, nx_, ny_, σ_] := Module[{t}, t = 1/2 σ^2;
  GaussianDerivative[{t, nx}, {t, ny}][im]]
Protect[gD];
SetOptions[RasterPlot, Frame -> False];
```

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7.1 Limits on differentiation: scale, accuracy and order

For a given order of differentiation we find that there is a limiting scale-size below which the results are no longer exact. E.g. when we study the derivative of a ramp with slope 1, we expect the outcome to be correct. Let us look at the *observed* derivative at the center of the image for a range of scales ($0.4 < \sigma < 1.2$ in steps of 0.1):

```
im = Table[x, {y, 64}, {x, 64}];
b = Table[{σ, gDf[im, 1, 0, σ]⟦32, 32⟧},
   {σ, 0.4`, 1.2`, 0.1`}];
ListPlot[b, Joined → True, PlotRange → All, AxesLabel →
  {"σ", "\!\(\*SubscriptBox[\(∂\), \(x\)]\)L"},
 PlotStyle → Thickness[0.01`], ImageSize → 250]
```

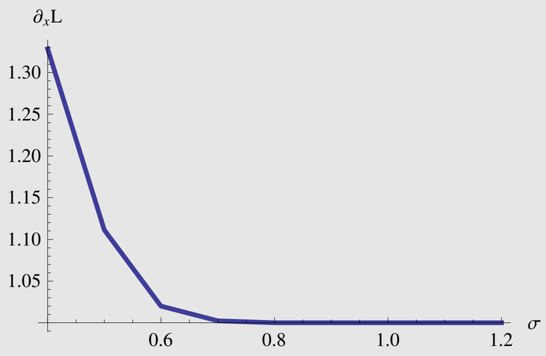

Figure 7.1 The measured derivative value of the function $y = x$ is no longer correct for decreasing scale. For scales $\sigma < 0.6$ pixels a marked deviation occurs.

There is a fundamental relation between the order of differentiation, scale of the operator and the accuracy required [TerHaarRomeny1994b]. We will derive this relation.

The Fourier transform of a Gaussian kernel is again a Gaussian:

```
Unprotect[gauss]; gauss[x_, σ_] := 1/Sqrt[2 π σ^2] Exp[-x^2/(2 σ^2)];
fftgauss[ω_, σ_] =
 Simplify[FourierTransform[gauss[x, σ], x, ω], σ > 0]
```

$$\frac{e^{-\frac{1}{2}\sigma^2\omega^2}}{\sqrt{2\pi}}$$

The Fourier transform of the derivative of a function is $-i\,\omega$ times the Fourier transform of the function:

```
FourierTransform[∂x gauss[x, σ], x, ω] /
 FourierTransform[gauss[x, σ], x, ω]
```

$-i\,\omega$

The Fourier transform of the n-th derivative of a function is $(-i\,\omega)^n$ times the Fourier transform of the function.

A smaller kernel in the spatial domain gives rise to a wider kernel in the Fourier domain, as shown below for a range of widths of first order derivative Gaussian kernels (in 1D):

```
GraphicsRow[
 {Plot3D[fftgauss[ω, σ], {ω, -π, π}, {σ, 0.4`, 2},
   PlotPoints → 30, AxesLabel → {"ω", "σ", "fft"},
   Axes → True, Boxed → True],
  Plot3D[gauss[x, σ], {x, -π, π}, {σ, 0.4`, 2},
   PlotPoints → 30, AxesLabel → {"x", "σ", "gauss"},
   Axes → True, Boxed → True]}, ImageSize → 490]
```

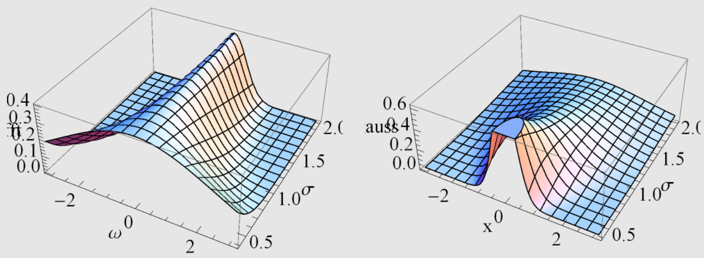

Figure 7.2 Left: The Fourier transform of the Gaussian kernel is defined for $\pi < \omega < \pi$. The function repeats forever along the frequency axis over this domain. For decreasing scale σ in the spatial domain the Fourier transform get wider in the spatial frequency domain. At some value of σ a significant leakage (aliasing) occurs. Right: The spatial Gaussian kernel as a function of scale.

We plot the Fourier spectrum of a kernel, and see that the function has signal energy outside its proper domain $[-\pi, \pi]$ for which the spectrum is defined:

```
Plot[{If[-π < ω < π, fftgauss[ω, 0.5`], 0],
   fftgauss[ω, 0.5`]}, {ω, -2 π, 2 π},
 Filling → Axis, Ticks → {{-π, π}, Automatic},
 AxesLabel → {"ω", "𝒢(ω,σ=.5)"}, ImageSize → 350]
```

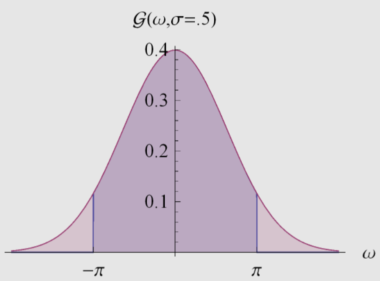

Figure 7.3 The definition of the leakage is the (unshaded) area under the curve outside the definition domain, relative to the total area under the curve. Here the definition is given for the 1D Gaussian kernel. Due to the separability of Gaussian kernels this definition is easily extended to higher dimensions.

The error is defined as the amount of the *energy* (the square) of the kernel that is 'leaking' relative to the total area under the curve (note the integration ranges):

```
error[n_, σ_] = Simplify[100 (∫_π^∞ (I ω)^(2 n) fftgauss[ω, σ]^2 dω) / (∫_0^∞ (I ω)^(2 n) fftgauss[ω, σ]^2 dω),
  {Re[σ^2] > 0, Re[n] > -1/2}]
```

$$\frac{100\ \text{Gamma}\left[\frac{1}{2}+n,\ \pi^2\ \sigma^2\right]}{\text{Gamma}\left[\frac{1}{2}+n\right]}$$

We plot this Gamma function for scales between $\sigma = 0.2 - 2$ and order of differentiation from 0 to 10, and we insert the 5% error line in it:

```
p1 = Plot3D[error[n, σ],
  {σ, .2, 2}, {n, 0, 10}, PlotRange → All,
  AxesLabel → {"σ", "n", "error %  "}, Boxed → True,
  Axes → True, Exclusions → {error[n, σ] == 5},
  ExclusionsStyle → {Red}]
```

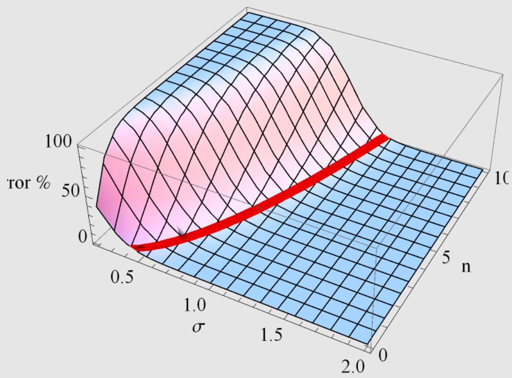

Figure 7.4 Relation between scale σ, order of differentiation n, and accepted error (in %) for a convolution with a Gaussian derivative function, implemented through the Fourier domain. The red line marks the 5% error.

```
ContourPlot[error[n, σ], {n, 0, 10}, {σ, 0.1`, 2},
 ContourShading → False, Contours → {1, 5, 10},
 ContourStyle -> Directive[{Thick, Red}],
 FrameLabel → {"Order of differentiation",
   "Scale in pixels"}, ImageSize → 275]
```

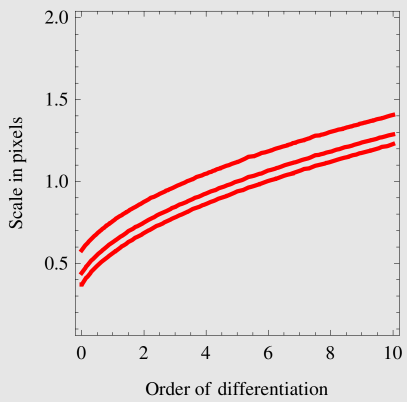

Figure 7.5 Relation between scale σ and the order of differentiation n for three fixed accuracies for a convolution with a Gaussian derivative function, implemented through the Fourier domain: upper graph: 1%, middle graph: 5%, lower graph: 10% accepted error.