MSc TU/e Course:

Differential Geometry for Image Processing

Teachers:

- R. Duits MF 7.072 (responsible teacher, lecturer)
- E.J. Bekkers MF 7.074 (teacher, instructor)

Course Description:

This MSc course aims to provide applied mathematicians and mathematical engineers with advanced differential geometrical knowledge and with effective multi-orientation algorithms. It consists of advanced differential geometry (60%), geometric scientific computing (20%) and (industrially oriented) medical imaging applications (20%). The course is theoretically oriented, does not require pre-knowledge on image processing, nor does it require a high expertise on the programming language *Mathematica (Wolfram-Research).* There will be self-contained notebooks that illustrate the theory for hands-on experience with modest practical assignments.

The main topics are:

1. Locally Adaptive Frames in R^d (that are adapted to images),
2. Multi-Orientation Representations of Images (known as "orientation scores"),
3. Continuous Wavelet Transforms,
4. Lie groups, with a particular focus on roto-translation SE(d) ("Special Euclidean Motion Group"),
5. Partial Differential Equations on SE(d),
6. Optimal Geometric Control via Finsler geometry and Ordinary Differential Equations on SE(d),
7. Locally Adaptive Frames in SE(d) (that are adapted to orientation scores).

Although these topics are useful in many applications (optimal control, robotics, physics, mechanics, cortical modeling, crowd-dynamics, image analysis) we illustrate the strength of our theory only on geometric medical imaging applications. We consider:

- complex vasculature tracking in medical images (X-ray, optical images),
- enhancement of crossing fibers in medical images (optical images, DW-MRI),
- connectivity quantification in medical images (DW-MRI).

In these medical imaging applications reduction of radiation dose and acquisition time leads to low-contract, noisy medical images, which make these tasks challenging. Moreover, the elongated structures in these images (e.g. vasculature, neural fibers) exhibit crossings and bifurcations. Where conventional tracking and enhancement techniques in the image domain fail, our differential geometrical *multi-orientation* algorithms provide the right tools for tracking and enhancement of complex line structures in these medical images.

Course Material:

Our own lecture notes cover everything in chronological order.

Schedule:

5-ECTS to be scheduled in 4th quartile.
Course Type: graduate school.
Academic year: 2018-2019 (Capita Selecta)
Academic year: 2019 onwards (MSc TU/e Course)

Setup:

- 7 x 4 hours lectures.
- 7 x 2 hours instructions.

Target Audience:

Master Students of Industrial and Applied Mathematics (IAM).

Also for mathematically interested and talented (MsC or PhD)-students from the following departments:
- Physics,
- Electrical Engineering/Computer Science,
- Biomedical Engineering,
- Mechanical Engineering.

Pre-knowledge:

The course is mainly theoretically oriented and does not require image processing pre-knowledge, nor does it require high expertise on programming language *Mathematica (Wolfram-Research)*.
It does require a **strong background** in:
analysis, vector calculus, linear algebra, Fourier theory and partial differential equations (PDEs).
It requires a **basic background** in:
functional analysis and scientific computing (of PDEs).

Courses (or similar) that are considered necessary (in order of relevance):

- Analysis & Calculus (for example: Bachelor Courses 2WA40, 2WA60, 2DBN10)
- Partial differential equations (Bachelor Course, 2WA90)
- Linear Algebra (for example: both Bachelor Courses 2WF30, 2DNN0)
- Applied functional analysis (Master Course, 2MMA10, Q2)

Courses that are considered optional and beneficiary (in order of relevance):

- Tensor calculus and differential geometry (2WAH0, Q3)
- Front-End Vision (8D010, November 2018, Q2)
- Scientific Computing (2MMN10, Q1)
- Probability and Stochastics (2MMS10, Q1)
- Evolution equations (2MMA40, Q4)

Detailed Schedule:

(Title - Content – Exercise Topics – Learning Objectives- per Lecture)

Lecture 1

General introduction to PDE-based image processing.

Content:

- Differentiation of images and vector fields on R^d.
- Multi-scale representations of images.
- Linear Scale space representations on the unbounded domain and bounded domain and their exact solutions.
– Gaussian derivatives.
-Image regularization techniques and their relation.
– Understand practical image representations and interpolations via B-splines.

Exercise Topics:

- Solving PDE's for linear scale spaces on the unbounded domain and bounded domain.
- Analyzing topological transitions in scale space representations.
- Well-posed differentiation of images: Gaussian Derivatives.
- Separable operators in Gaussian scale space representations.
- Gaussian derivatives and the Hermite basis for images.
- Image interpolation and representation via B-splines.
- Tikhonov regularization and its connection to diffusion by the Laplace transform.

Learning Objectives:

- Master techniques to solve linear scale space PDE's on the unbounded and bounded domain.

- Being able to compute operator norms of Gaussian Derivatives.

- Understand the Laplace transform relation between linear diffusion and Tikhonov regularization

(and understand the underlying probabilistic interpretation)

- Being able to derive Euler-Lagrange equations for basic variational methods (e.g. Tikhonov regularization).

- Understand image representations and interpolations via B-splines and being able to apply them via prepared *Mathematica* notebooks

Course Material Lecture 1:

Lecture notes: ***Part I*** *(ch: 1, 2, 4, 5 & ch: 9.1)*

Lecture 2

Locally adaptive frames in image processing, their shortcomings, and the motivation for Orientation Scores.

Content:

- Nonlinear diffusions with data-driven diffusivity (scalar-diffusivity or matrix-diffusivity).
- Locally adaptive frames based on the structure tensor of the image.
- Locally adaptive frames based on the Hessian of the image.
- Locally adaptive frames for geometrical image processing
  (via nonlinear PDE's and differential operators).
- Problems with Locally adaptive frames directly in image domain.
- Understand variational techniques (total variation flow) for image regularization/denoising, and their relation to nonlinear diffusions with scalar diffusivity.

Exercise Topics:

- Locally adaptive frames via the Hessian of the image & corresponding optimization.
- Locally adaptive frames via the Structure tensor of the image & corresponding optimization.
- Vector fields on manifolds: the algebraic and geometric viewpoint.
- Data-driven diffusions via locally adaptive frames.
- Observe the limitations and problems of locally adaptive frames directly in image domain.
- Reformulating the optimization problems by exponential curve fits of the 1^{st} and 2^{nd} kind, such that it allows generalization to other Lie groups.
- Analysis of image regularization by total variation flows.

Learning Objectives:

- Master geometric and algebraic treatment of vector fields on manifolds, and know how they relate.

- Understand the optimization problems and Euler-Lagrange formalisms underlying locally adaptive frames, and know how to put them in a more general Lie group formalism.

- Understand possibilities and limitations of locally adaptive frames directly in the image domain.

- Know how to apply variants of nonlinear images diffusions and total variation flows via prepared *Mathematica* notebooks

- Know how to analyze Total Variation regularization.

Course Material Lecture 2:

*Lecture notes: **Part I** (ch: 6-8 & ch: 9.2):*

Extra Literature on topics of Lecture 2 (not part of the course & exam):

- "Chapter: Differential Structure in Book "Front-End Vision" by Prof. B.M. ter Haar Romeny" see http://bmia.bmt.tue.nl/Education/Courses/FEV/course/notebooks/Examples%20of%20Differential%20invariant%20structure%20MMA10.nb
- Chapter 1 and Chapter 4 in http://bmia.bmt.tue.nl/people/RDuits/1502.08002v3.pdf

https://link.springer.com/article/10.1023/A:1008009714131

Lecture 3 (invertible multi-orientation image representations)

Introduction to Image Processing via Invertible Orientation Scores.

Content: In this lecture we aim for introduction and motivation of the orientation score framework, where we answer the following questions:

- What is an invertible orientation score?
- Why must we apply left-invariant processing on them?
- What medical image analysis applications can we tackle with them?
- How can we quantify the stability of the transform between image and orientation score?
- Can we obtain general unitarity results of transformations between images and scores?

Exercise Topics:

- Condition numbers of the transformation.
- Representation Theory, and symmetries of the transformation.
- Proofs of general unitarity results.
- Reproducing kernel Hilbert spaces.
- the Lie group structure on the domain of an orientation score.

Learning Objectives:

-Understand the general benefits of an invertible orientation score.

-Understand that the domain of an orientation score of a 2D image is the Lie group SE(2).

-Understand that the domain of an orientation score of a 3D image is Lie group quotient $SE(3)/(0 \times SO(2))$.

-Understand the notion of group representations and know how they enter both the construction and the processing of orientation scores.

-Be able to show that orientation score processing must be left-invariant and not right-invariant.

-Know how to prove general unitarity results of transforms between images and scores.

-Know how to quantify the stability of the transform between an image and an orientation scores.

Course Material Lecture 3:

*Lecture notes; **Part II**: ch: 1 & 2*

Extra Literature on topics of Lecture 2 (not part of the course & exam):

http://bmia.bmt.tue.nl/people/RDuits/IJCV2007.pdf
http://erikbekkers.bitbucket.io/data/pdf/Bekkers_PhD_Thesis.pdf (2016, section 2 only)
http://bmia.bmt.tue.nl/people/RDuits/THESISRDUITS.pdf (2005, section 4 only)
http://bmia.bmt.tue.nl/people/RDuits/qampartI.pdf (2010, section 1 & 2 only)

Lecture 4 (design of multi-orientation image representations)

***Design of Invertible Orientation Scores of 2D and 3D images*.**

Content: In this lecture we focus on the following topics:

- Design of 2D Cake-wavelets.
- Design of 3D Cake-wavelets.
- Definition proper wavelet and proper wavelet sets.
- References to Tutorials of the Lie Analysis Mathematica 11 Package.

Exercise Topics:

- Design of 2D and 3D cake-wavelets in the Fourier domain.
- Proper wavelets.
- Proper wavelets with analytic description in spatial and Fourier domain.
- *Mathematica* Notebooks where you can play with cake-wavelet design in our Lie Analysis package. **www.lieanalysis.nl**

Learning Objectives:

- Understand the design of invertible orientation scores.
- Understand the design of proper wavelets.
- Be able to prove that a wavelet is proper or not.

- Know how to apply invertible orientation scores to images via the 'Lie-Analysis' *Mathematica* package that you can download from **www.lieanalysis.nl**

Course Material Lecture 4:

*Lecture notes; **Part II**: ch: 3 & 4*

Extra Literature on topics of Lecture 3 (not part of the course & exam):

http://bmia.bmt.tue.nl/people/RDuits/qampartI.pdf (2010, appendix A only)
http://bmia.bmt.tue.nl/people/RDuits/InvertibleOrientationScores.pdf (2015)
http://bmia.bmt.tue.nl/people/RDuits/InvertibleOrientationScoresof3DimagesComplete.pdf (2017)
http://bmia.bmt.tue.nl/people/RDuits/rana13-17.pdf (2015)
http://bmia.bmt.tue.nl/people/RDuits/CEDOS-3D.pdf (2017)

Lecture 5 & 6 (tracking via `shortest curves' in orientation scores)

Globally Optimal Sub-Riemannian geodesics in the projective line bundle.
Improvements via Asymmetric Finsler geometry: key-points instead of cusps.

Content: In the 5th and 6th lecture we give mathematical explanations and applied illustrations of geodesic tracking in orientation scores. We consider:

- Methods and theory to compute and analyze minimizing geodesics on manifolds.
- Sub-Riemannian / Sub-Finslerian Geodesics in SE(2) and 2D vessel-tracking applications.
- Sub-Riemannian / Sub-Finslerian Geodesics in SE(3) and 3D fiber tracking applications.
- The problem of cusps in spatial projections of SR geodesics in SE(2).
- The solution to the problem of cusps: Turn off the reverse gear of the Reeds-Shepp car.

Exercise Topics:

- Tangent bundles, Finsler functions and quasi-distances above the base manifold of positions and orientations (embedded as a Lie group quotient in SE(d)).
- Analyze the Hamiltonian equations for sub-Riemannian geodesics in SE(2).
- Preservation laws and symmetries of sub-Riemannian geodesics in SE(2).
- Analyze sub-Riemannian spheres on SE(2) and their intersection with the 1st Maxwell set.
- *Mathematica* notebooks where you can apply a vessel-tracking in a challenging industrial medical imaging example.

Learning Objectives:

- Understand the notion of a tangent bundle above the base manifold of positions and orientations.
- Understand the notion of a Finsler function and a metric tensor on such a tangent bundle.
- Understand how the Finsler function induces a (quasi-)distance and curve optimization problem.
- Understand the canonical Hamiltonian equations for curve optimization problems
- Understand techniques to derive the geodesics analytically.
- Understand the basic principles of numeric algorithms that solve data-driven versions of the model
- Know how to apply vessel-tracking in industrial medical imaging examples, and understand how the theoretical concepts enter the image analysis applications.

Course Material Lecture 5 & 6: *Lecture Notes* ***part III***

Extra Literature on topics of Lecture 5 &6 (not part of the course & exam):

(JMIV 2018, Duits, Meesters, Mirebeau, Portegies)
http://www.sciencedirect.com/science/article/pii/S092842570300072X (2003, Petitot: SR-cortical models)
http://link.springer.com/article/10.1007/s10851-005-3630-2 (2006, Citti-Sarti: SE(2)-SR-Geodesic cortical model)

ftp://ftp.botik.ru/rented/CPRC/www/sachkov/max_sre2_COCV.pdf (2009, Sachkov-Moiseev: SE(2)-SR-Geodesic solutions &Maxwell sets)

http://www.numdam.org/item/COCV_2011__17_2_293_0 (2011, Sachkov: Optimal synthesis and cut locus SE(2)-SR-Problem)

http://www.lsis.org/rossif/articles/Suzdal08.pdf (2010, Boscain, Charlot & Rossi: the problem of cusps)
http://bmia.bmt.tue.nl/people/RDuits/cusp.pdf (2014, SE(2)-SR-Geodesic model: Analysis of cusps in spatial projections SR-geodesics and geometric properties)

http://bmia.bmt.tue.nl/people/RDuits/siims.pdf (2015, SE(2)-SR-Geodesic model: Data-driven global minimizers)

(2016, PhD thesis Erik Bekkers)

http://bmia.bmt.tue.nl/people/RDuits/Paper1_81_REMCO.pdf (2016, SE(3)-SR-Geodesics solutions)
http://bmia.bmt.tue.nl/people/RDuits/GSI2017.pdf (2017, SR-geodesics and 1st Maxwell sets on projective line bundle)

http://bmia.bmt.tue.nl/people/RDuits/main_FM_full.pdf (2017, full survey minimizing geodesics in position and orientation space)

Lecture 7 (crossing-preserving diffusion via `straight curves' in orientation scores)

Crossing-preserving diffusions via locally adaptive frames in invertible orientation scores.

Content: In this lecture we focus on the following topics:

- Exponential maps and exponential curves in Lie groups.
- Exponential Curve fits in Lie group SE(d) of order 1.
- Exponential Curve fits in Lie group SE(d) of order 2.
- Crossing-Preserving Diffusions of images.
- The minus Cartan connection: "Straight curves" vs. "shortest curves".

Exercise Topics:

- Analytic computation of exponential curves in low-dimensional Lie groups like SE(2).
- Derive Euler-Lagrange equations for optimization in exponential curve fits.
- Construction of locally adaptive frames in SE(d) from exponential curve fits.
- Connections: Levi-Civita connection and the (minus) Cartan Connection on SE(d).
- *Mathematica Notebooks* with Medical imaging applications of crossing-preserving diffusions.

Learning Objectives:

- Know how to compute exponential curves in low dimensional Lie groups such as SE(2).
- Know how to compute exponential curve fits in SE(2) of the 1st order .
- Know how to compute exponential curve fits in SE(2) of the 2nd order.
- Understand the concept of a connection on the tangent bundle of a Lie group.
- Understand the role of the minus Cartan connection in
  * the exponential curve fit problem.
  * the geodesic optimizations of Lecture 5 & 6.
- See how crossing-preserving diffusions work on medical imaging examples in *Mathematica*, and understand how the theoretical concepts enter the applications.

Course Material Lecture 7: *Lecture Notes* ***part IV*** *and* ***Appendix D of part III.***

Extra Literature on topics of Lecture 7 (not part of the course & exam):

http://bmia.bmt.tue.nl/people/RDuits/1502.08002v3.pdf (2016)
http://erikbekkers.bitbucket.io/data/pdf/Bekkers_PhD_Thesis.pdf (2016, ch:4.5, ch:11)
https://www.sciencedirect.com/science/article/pii/S0926224517300852 (DGA 2017)
http://bmia.bmt.tue.nl/people/RDuits/Final_1411nm.pdf (chapter 5, NM-TMA 2016)
http://bmia.bmt.tue.nl/people/RDuits/qampartI.pdf (QAM AMS 2010)
http://bmia.bmt.tue.nl/people/RDuits/Duits-VanAlmsick.pdf (QAM AMS 2008)
http://bmia.bmt.tue.nl/people/RDuits/SE3JorgPortegiesDuits.pdf
http://bmia.bmt.tue.nl/people/RDuits/SSVMNorwaySummarySE2.pdf (2009)
The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups (Agrachev, Boscain, Gauthier, Rossi, 2009)
http://www.win.tue.nl/analysis/reports/rana05-43.pdf (2005)