Journal of Mathematical Imaging and Vision 20: 267–298, 2004

On the Axioms of Scale Space Theory

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Abstract. We consider alternative scale space representations beyond the well-established Gaussian case that satisfy all "reasonable" axioms. One of these turns out to be subject to a *first order* pseudo partial differential equation equivalent to the Laplace equation on the upper half plane $\{(x, s) \in \mathbb{R}^d \times \mathbb{R} \mid s > 0\}$. We investigate this so-called Poisson scale space and show that it is indeed a viable alternative to Gaussian scale space. Poisson and Gaussian scale space are related via a one-parameter class of operationally well-defined intermediate representations generated by a fractional power of (minus) the spatial Laplace operator.

Keywords: Gaussian scale space, Poisson scale space, α scale spaces, scale space axiomatics, semigroup theory

1. Introduction

Constructions of linear scale space representations based on rigorous axiomatics date back to the 1960's with the publications by Iijima [22, 23] in the context of pattern recognition. This early work was followed by numerous publications [1, 11, 19, 24–30, 36]. The pivot in all derivations is some set of axioms expressing desirable group and/or semigroup properties. For an overview cf. Weickert [38].

It is commonly taken for granted that the Gaussian scale space paradigm is the unique solution to a set of reasonable axioms if one disregards minor modifications, such as spatial inhomogeneities [12], diffeomorphisms [13, 14], and anisotropies [37], which can be easily accounted for. That this is in fact not true has been pointed out by Pauwels [30], who proposed a one-parameter class of scale space filters in Fourier space, which are compatible with some basic axioms.[1] Under the assumption of positivity the corresponding parameter domain is a finite interval $\alpha \in (0, 1]$, where $\alpha = 1/2$ and $\alpha = 1$ the correspond to Poisson, respectively Gaussian scale space.

In this article we scrutinize properties of these α scale spaces (and in particular Poisson scale space) in the spatial domain, and show that they indeed obey *all* basic axioms initially believed to hold only for the Gaussian case.[2] To demonstrate this we adopt an overcomplete set of axioms that capture the various subsets that have been employed in the derivation of the Gaussian scale space paradigm. In addition, some of the conjectures raised by Pauwels [30] are verified by rigorous proofs, whereas some intuitive expectations are disproved. For example, it turns out that, contrary to previous belief, the Fourier filters for *all* parameter values, including the Poisson filter, do possess infinitesimal generators in the spatial domain in the sense of linear derivative operators.

The most natural case of all α scale spaces is the Poisson scale space since it is the only one where the scale parameter has the same physical dimension as the spatial variables x_i, allowing Euclidean geometry *within scale space*. Moreover, its Clifford analytic extension, which is first introduced in a computer vision context by Felsberg and Sommer [9] has several practical benefits. Initially, Felsberg and Duits worked independently on the subject of Poisson scale space following a different approach, but recently a cooperation has started on the subject of finite domain scale spaces, cf. [5, 8] which will not be considered in this article. In this article Poisson scale space is shown to be associated with the *first order linear scale space pseudo p.d.e.*

$$\frac{\partial u}{\partial s} = -\sqrt{-\Delta}\, u,$$

which at the same time illustrates the cause of previous failure, viz. the fact that the possibility of a generator in the form of a *fractional power of a derivative operator*—the precise definition of which will be outlined in this article—has been overlooked. The solution of this equation for initial condition f is nevertheless a perfectly viable, smooth scale space image, which is essentially different from its Gaussian counterpart. The solution can in fact be written as a straightforward convolution in closed-form using Poisson filters. Just like filtering with Gaussian kernels is equivalent to solving the diffusion equation on the upper half space, filtering with Poisson kernels is equivalent to solving the Dirichlet problem on the upper half space. Finally, we establish an explicit connection between the various scale space representations, in particular Poisson and Gaussian scale spaces. Although we will not focus explicitly on probability theory in this article, we would like to mention that α-scale spaces correspond to α-stable Lévy motions in the field of stochastic[3] processes [34].

2. Preliminaries and Notation

A point in scale space $\mathbb{R}^d \times \mathbb{R}^+$ will be denoted by $(\mathbf{x}, s)$. Sometimes (if scale is fixed) we will write $\hat{\mathbf{x}}$ for short. Let $\mathbf{a} \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ and define Σ_λ, S_λ, $T_\mathbf{a}$ by

$$\begin{aligned}
[\Sigma_\lambda f](\mathbf{x}) &= f(\mathbf{x}/\lambda)\\
[S_\lambda f](\mathbf{x}) &= \lambda f(\mathbf{x})\\
[T_\mathbf{a} f](\mathbf{x}) &= f(\mathbf{x}-\mathbf{a})
\end{aligned} \tag{1}$$

Let Ω be a subset of $\mathbb{R}^d$, then $\bar{\Omega}$ denotes the closure of Ω. The boundary of Ω which equals $\bar{\Omega} \setminus \Omega$ will be denoted by $\partial\Omega$. The outward normal to the boundary will be written $\mathbf{n}$. The d-dimensional ball in $\mathbb{R}^d$ with respect to the Euclidean metric, with center $\mathbf{a}$ and radius $R > 0$, will be denoted by $B_{\mathbf{a},R}$. The total surface measure of $\partial B_{\mathbf{a},R}$ equals $\sigma_d R^{d-1}$. From $\int_{\mathbb{R}^d} e^{-\|\mathbf{x}\|^2} d\mathbf{x} = \pi^{d/2}$, it easily follows that

$$\sigma_d = \frac{2\,(\pi)^{\frac{d}{2}}}{\Gamma(d/2)}. \tag{2}$$

In the 1D case, $d = 1$, we occasionally use complex function theory. We then use the following notation $z = x + is = re^{i\theta}$. We denote the real respectively imaginary part of a complex number w by $\Re(w)$ respectively $\Im(w)$.

With regard to spaces we use the following notation:

- $\mathcal{L}(X, Y)$ = the vector space consisting of linear operators from X into Y. If $X = Y$ we write $\mathcal{L}(X)$ for short.
- $\mathcal{B}(X, Y)$ = the vector space consisting of continuous linear operators from X into Y. If $X = Y$ we write $\mathcal{B}(X)$ for short.
- If $\mathcal{A} \in \mathcal{B}(X)$, with spectrum $\sigma(\mathcal{A})$, then its resolvent is defined by

  $$R(\lambda; \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}.$$

  for all $\lambda \in \rho(\mathcal{A}) = \mathbb{C} \setminus \sigma(\mathcal{A})$.
- The dual of a vector space X is the vector space consisting of all continuous linear functionals on X and will be denoted by X'.
- $\mathbb{L}_p(\Omega, \mu)$ = the quotient space consisting of functions with finite L_p norm $(p > 0)$, i.e. $(\int_\Omega |f|^p \,\mathrm{d}\mu)^{1/p} < \infty$ on Ω with respect to the nil space of the L_2-norm, which consists of all functions f on Ω with zero measure support, i.e. $f = 0$ almost everywhere. Mostly μ equals the usual Lebesgue measure $\mathrm{m_d}$ and then we write $\mathbb{L}_p(\Omega)$ for short.
- The Fourier transform $\mathcal{F} : \mathbb{L}_2(\mathbb{R}^d) \to \mathbb{L}_2(\mathbb{R}^d)$, is (A.E.) defined by

  $$[\mathcal{F}(f)](\omega) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\mathbf{x})\, e^{-i\omega\cdot\mathbf{x}}\, d\mathbf{x},$$

  mostly we will write $\hat{f}$ in stead of $\mathcal{F}(f)$.
- The Laplace transform[4] $\mathcal{L}(f)$ of a function $f \in \mathbb{L}_2(\mathbb{R}^+)$ is defined by

  $$[\mathcal{L}(f)](\lambda) = \int_0^\infty f(x)\, e^{-\lambda x}\, dx,$$

  for $\Re(\lambda) > 0$.
- Associate to each $s > 0$ a positive measure μ_s, by setting

$$\mathrm{d}\mu_s(\mathbf{y}) = (1 + \|\mathbf{y}\|^2)^s \,\mathrm{dm}_d(\mathbf{y}) \quad s > 0 \tag{3}$$

- $C^n(\Omega)$ = the vector space consisting of all n times continuous differentiable functions on Ω.
- $\mathcal{D}(\Omega)$ = the vector space consisting of all infinitely differentiable functions with compact support within Ω. This space is equipped with the local convex topology generated by the semi-norms $q_N : \mathcal{D}(\Omega) \to \mathbb{R}$ $(N \in \mathbb{N})$ given by

$$q_N(f) = \sup_{|\alpha|<N} \sup_{\|\mathbf{x}\|<N} |(D^\alpha f)(\mathbf{x})|$$

- Define the semi-norms $p_N : C^\infty(\mathbb{R}^d) \to \mathbb{R}$ $(N \in \mathbb{N})$ by

$$p_N(f) = \sup_{|\alpha|<N} \sup_{\mathbf{x}\in\mathbb{R}^d} (1 + \|\mathbf{x}\|^2)^N |(D^\alpha f)(\mathbf{x})|.$$

- $\mathcal{S}_d$ = the vector space consisting of all infinitely differentiable function f such that $p_N(f) < \infty$ for all $N \in \mathbb{N}$. This space is equipped with the local convex topology generated by the semi-norms p_N. The elements in the dual space $\mathcal{S}'_d$ are often called tempered distributions. The Fourier transform of a tempered distribution Λ is defined[5] by $\hat{\Lambda}(\phi) = \Lambda(\hat{\phi})$.
- $\mathbb{H}_s(\mathbb{R}^d)$ equals the vector space consisting of all tempered distributions Λ which Fourier transform is in $\mathbb{L}_2(\mathbb{R}^d; \mu_s)$.

The kernels $G_s : \mathbb{R}^d \to \mathbb{R}$, $H_s : \mathbb{R}^d \to \mathbb{R}$ are defined by

$$\begin{aligned} G_s(\mathbf{x}) &= \frac{1}{(4\pi s)^{d/2}} e^{-\frac{\|\mathbf{x}\|^2}{4s}} \\ H_s(\mathbf{x}) &= \frac{2}{\sigma_{d+1}} \frac{s}{(s^2 + \|\mathbf{x}\|^2)^{\frac{d+1}{2}}}. \end{aligned} \tag{4}$$

Note that G_s is a rapidly decreasing function while H_s is not. This means that the distributional approach to scale space theory cf. [11], regarding raw images as distributions defined on the test space of rapidly decreasing functions, needs some adaptation for other semigroups such as Poisson filtering (see Appendix).

The mapping from the original image f and scale s onto the blurred image will be denoted by $\Phi : \mathbb{L}_2(\mathbb{R}^d) \times \mathbb{R}^+ \to \mathbb{L}_2(\mathbb{R}^d)$. The blurred image at a fixed scale $s > 0$ will be denoted by u and is given by

$$u(\mathbf{x}, s) = \Phi[f, s](\mathbf{x}), \mathbf{x} \in \mathbb{R}^d$$

In order to stress that $s > 0$ is fixed, we will often write $\Phi_s f$ in stead of $\Phi[f, s]$.

3. Axioms

First we will summarize some basic observations with respect to blurring:

- If the scale s tends to zero, the blurred image must tend to the original image. For continuous images this convergence must be pointwise.
- If two images f_1, f_2 satisfy $f_1(\mathbf{x}) \leq f_2(\mathbf{x})$ almost everywhere on $\mathbb{R}^d$, then the corresponding blurred images u_1, u_2 must satisfy $u_1(\mathbf{x}, s) \leq u_2(\mathbf{x}, s)$ almost everywhere on $\mathbb{R}^d$ for all $s > 0$.
- Successive blurring at scale $s_1 > 0$ and $s_2 > 0$ must correspond to a single, effective blurring with an aperture $s > 0$ uniquely determined by s_1 and s_2. In this report we focus on the case

$$s = s_1 + s_2 \tag{5}$$

  Note that the cases $s = (s_1^p + s_2^p)^{1/p}$, with $0 < p < \infty$ can be brought to (5) after re-parameterizing according to $s' = s^p$.
- There are two ways of imposing causality constraints:
  - *Weak causality*: Local extrema with respect to both scale ($s > 0$) and space ($\mathbf{x} \in \mathbb{R}^d$) within scale space are not allowed: Closed isophotes within scale space are not allowed.
  - *Strong causality*: Blurring an image must lead to less extreme grey values. Local extrema with respect to space (not scale) should not enhance.
- Blurring an image should be an isotropic process, since a priori we do not know the internal structure of an image.
- Blurring a translated image is the same as translating the blurred image.
- The operator which maps an original image onto its blurred image on a fixed scale, will be assumed to be linear.
- During the blurring process information will be lost. So, from an information theoretical point of view entropy should increase during the blurring process.

These requirements will be formalized as follows:

1. An arbitrary original image f is assumed to be a member of $\mathbb{L}_2(\mathbb{R}^d)$ with compact support.

2. For all $f \in \mathbb{L}_2(\mathbb{R}^d)$ we must have

$$\Phi[T_{\mathbf{a}} f, s] = T_{\mathbf{a}} \Phi[f, s]$$

(i.e. translation invariance).

3. For all $\lambda > 0$ and $s > 0$ there exists a unique s' such that for all $f \in \mathbb{L}_2(\mathbb{R}^d)$

$$\Phi[\Sigma_\lambda f, s] = \Sigma_\lambda \Phi[f, s']$$

We will assume that the corresponding rescaling function Ψ which maps s onto s' is a strictly increasing continuous function such that $\Psi(0) = 0$ and $\Psi(s) \to \infty$ as $s \to \infty$.

4. Preservation of positivity:

$$f \geq 0 \Rightarrow \Phi[f, s] \geq 0.$$

5. The blurring operator $f \mapsto \Phi[f, s]$ ($s > 0$ fixed) can be regarded in two different ways

I assume that operator $f \mapsto \Phi[f, s] \in \mathcal{B}(\mathbb{L}_2(\mathbb{R}^d), \mathbb{L}_2(\mathbb{R}^d))$, i.e.

$$\Phi[f + g, s] = \Phi[f, s] + \Phi[g, s]$$
$$\Phi[S_\lambda f, s] = S_\lambda \Phi[f, s]$$

for all $s > 0$, $f, g \in \mathbb{L}_2(\mathbb{R}^d)$, and there exists a $C > 0$ such that

$$\|\Phi[f, s]\|_{\mathbb{L}_2(\mathbb{R}^d)} \leq C \|f\|_{\mathbb{L}_2(\mathbb{R}^d)}.$$

for all $f \in \mathbb{L}_2(\mathbb{R}^d)$ and $s > 0$ (fixed).

II assume that operator $f \mapsto \Phi[f, s] \in \mathcal{B}(\mathbb{L}_2(\mathbb{R}^d), \mathbb{L}_\infty(\mathbb{R}^d))$,

$$\Phi[f + g, s] = \Phi[f, s] + \Phi[g, s]$$
$$\Phi[S_\lambda f, s] = S_\lambda \Phi[f, s]$$

for all $s > 0$, $f, g \in \mathbb{L}_2(\mathbb{R}^d)$, and there exists a $C > 0$ such that

$$\|\Phi[f, s]\|_{\mathbb{L}_\infty(\mathbb{R}^d)} \leq C \|f\|_{\mathbb{L}_2(\mathbb{R}^d)}.$$

6. For all $s_1, s_2 > 0$ we must have

$$\Phi[\Phi[\cdot, s_1], s_2] = \Phi[\cdot, s_1 + s_2].$$

7. Causality constraints:

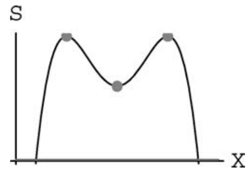


Figure 1. Weak causality.

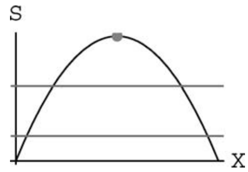


Figure 2. Strong causality.

(a) *Weak Causality Constraint*: Any scale space isophote $u(\mathbf{x}, s) = \lambda$ is connected to the ground plane, i.e. it is connected to a point $u(\mathbf{x}, 0) = \lambda$.

(b) *Strong Causality Constraint*: For every $s_1 \geq 0$ and $s_2 > 0$ with $s_2 > s_1$ the intersection of any connected component of an isophote within the domain $\{(\mathbf{x}, s) \in \mathbb{R}^d \times \mathbb{R}^+ \mid \mathbf{x} \in \mathbb{R}^d, s_1 \leq s < s_2\}$ with the plane $s = s_1$ should not be empty.

8. For all $f \in \mathbb{L}_2(\mathbb{R}^d)$ we must have

$$\lim_{s \downarrow 0} \Phi[f, s] = f \text{ in } \mathbb{L}_2 \text{ sense.}$$

Moreover if f is continuous, then the above limit also holds pointwise. If we restrict Φ to subspace $\mathcal{D}(\mathbb{R}^d) \times \mathbb{R}^+$ then we can write

$$\lim_{s \downarrow 0} \Phi[\cdot, s](\mathbf{x}) = \delta_{\mathbf{x}}$$

according to the weak star topology on $\mathcal{D}'(\mathbb{R}^n)$.

9. Rotation invariance, i.e. Let $R \in SO(d)$. Define $\mathcal{P}_R : \mathbb{L}_2(\mathbb{R}^d) \to \mathbb{L}_2(\mathbb{R}^d)$ by

$$[\mathcal{P}_R \psi](\mathbf{x}) = \Psi(R\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d.$$

Then we must have

$$\mathcal{P}_R \Phi_s[f] = \Phi_s[\mathcal{P}_R f],$$

for all $f \in \mathbb{L}_2(\mathbb{R}^d)$, $s > 0$.

10. Average grey-value invariance, i.e.

$$\|\Phi_s[f]\|_{\mathbb{L}_1(\mathbb{R}^d)} = \|f\|_{\mathbb{L}_1(\mathbb{R}^d)}, \tag{6}$$

for all $s > 0$, $f \geq 0 \in \mathbb{L}_1(\mathbb{R}^d)$.

11. *Increase of entropy*: Consider a scale space u of a positive image f, i.e. $u(\cdot, s) = \Phi[f, s]$ and $f > 0$ (and thereby by Axiom 4 we have $u > 0$) almost everywhere such that

$$[\mathcal{E}(u)](s) = -\int_{\mathbb{R}^d} u(\mathbf{x}, s) \ln u(\mathbf{x}, s)\, \mathrm{d}\mathbf{x} \quad (s > 0), \tag{7}$$

is finite. The mapping $\mathcal{E}(u) : \mathbb{R}^+ \to \mathbb{R}$ is called the entropy of u. Ignoring constants, the entropy is invariant scaling: $\mathcal{E}(\lambda u)(s) = \lambda \mathcal{E}(u)(s) + (\lambda \log \lambda)\, u_{av}$, where $\lambda > 0$, u_{av} is the average grey value of u. Using this scaling it is always possible to ensure that (7) is positive, since $u \log u < 0 \Leftrightarrow 0 < u < 1$. The entropy is a measure of missing information and therefore the entropy should be a monotone increasing function on[6] $\mathbb{R}^+$, i.e. $\frac{\partial}{\partial s}[\mathcal{E}(u)](s) > 0$ for all source images f. Moreover, we want $\frac{\partial}{\partial s}[\mathcal{E}(u)](s) \to 0$ if $s \to \infty$.

Next we summarize the direct consequences of these axioms. For instance by the following theorem it follows that $f \to \Phi[f, s]$, $(s > 0$ fixed) is an integral operator.

Theorem 1 (Dunford-Pettis). *Let X be a measurable space, according to measure $\mu : X \to \mathbb{R}^+$. Let $1 \leq p < \infty$. Let $\mathcal{A}$ be a bounded operator from $\mathbb{L}_p(X)$ into $\mathbb{L}_\infty(X)$, then there exists a $K \in \mathbb{L}_1(X \times X)$ such that*

$$\sup_x \left(\int_X |K(x, y)|^q \, d\mu(y) \right)^{1/q} = \|A\|,$$

with $q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and for all $f \in \mathbb{L}_p(X)$ we have that:

$$(\mathcal{A} f)(x) = \int_X K(x, y) f(y)\, d\mu(y)$$

for almost every $x \in X$.

For a proof see [3], pp. 113–114. If we take $X = \mathbb{R}^d$ equipped with the Lebesgue measure and $p = 2$, then[7] $q = 2$ and by Axiom 5(II) we have that mapping $f \mapsto \Phi[f, s]$, with $s > 0$ fixed, is an integral operator. By Axiom 2 (translation invariance) it follows that operator Φ is given by

$$\Phi[f, s](\mathbf{x}) = \int_{\mathbb{R}^d} K_s(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, \mathrm{d}\mathbf{y} \tag{8}$$

for a certain $K_s \in \mathbb{L}_1(X \times X)$.

Lemma 1. *If $f \in \mathbb{L}_1(\mathbb{R}^d)$, then $\hat{f} \in C_0(\mathbb{R}^d)$, and $\|\hat{f}\|_{\mathbb{L}_\infty(\mathbb{R}^d)} \leq (2\pi)^{-d/2} \|f\|_{\mathbb{L}_1(\mathbb{R}^d)}$*

For proof see [33], p. 169.

Notice that the mapping $f \mapsto K_s * f$ is also a continuous mapping from $\mathbb{L}_2(\mathbb{R}^d)$ into itself: By the Plancherel theorem Fourier Transform is a unitary operator on $\mathbb{L}_2(\mathbb{R}^d)$ and by the above lemma the Fourier Transform of K_s is continuous on $\mathbb{R}^d$ with $\|\hat{K}\|_{\mathbb{L}_\infty(\mathbb{R}^d)} < \infty$. So it follows that the mapping $f \mapsto \mathcal{F}(K_s * f) = \hat{K}_s \hat{f}$ is bounded on $\mathbb{L}_2(\mathbb{R}^d)$.

Next, we will use (8) in order to adjust the other axioms. Axiom 6 can now be written

$$K_{s_2} * (K_{s_1} * f) = K_{s_1 + s_2} * f, \quad \text{for all } f \in \mathbb{L}_2(\mathbb{R}^d). \tag{9}$$

Axiom 9 is satisfied if the Kernel K_s only depends on $\|\mathbf{x}\|$. Since, then we have $\mathcal{P}_R K_s = K_s$ and therefore

$$\begin{aligned}
\mathcal{P}_R \Phi_s[f](\mathbf{x}) &= \int_{\mathbb{R}^d} K_s(R\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, \mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^d} K_s(R(\mathbf{x} - \mathbf{u}) f(R\mathbf{u})\, \mathrm{d}R\mathbf{u} \\
&= \int_{\mathbb{R}^d} K_s(\mathbf{x} - \mathbf{u}) [\mathcal{P}_R f](\mathbf{u})\, \mathrm{d}\mathbf{u} \\
&= \Phi_s[\mathcal{P}_R f](\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^d
\end{aligned}$$

Axiom 4 is satisfied if and only if $K_s \geq 0$. An equivalent condition for the average grey value invariance axiom is that the $\mathbb{L}_1$-norm of the convolution kernel K_s equals 1, since for $f \geq 0$

$$\begin{aligned}
\|K_s * f\|_{\mathbb{L}_1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} K_s(\mathbf{y}) \left[\int_{\mathbb{R}^d} f(\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{x} \right] \mathrm{d}\mathbf{y} \\
&= \|f\|_{\mathbb{L}_1(\mathbb{R}^d)} \|K_s\|_{\mathbb{L}_1(\mathbb{R}^d)}.
\end{aligned}$$

It is shown by Pauwels [30] (for $d = 1$, but the generalization to arbitrary $d \in \mathbb{N}$ is straightforward) that if Axioms 1–3, 5, 6, 9, 10 are satisfied, the Fourier Transform of the convolution kernel *must* be equal to

$$K_s(x) = \frac{1}{\Psi(s)} \phi\left(\frac{x}{\Psi(s)} \right) \tag{10}$$

where the Fourier Transform of ϕ has the form:

$$\hat{\phi}(\omega) = e^{-a|\omega|^{2\alpha}} \quad \alpha > 0, a \geq 0$$

and the corresponding re-scaling function is given by $\Psi(s) = s^{1/(2\alpha)}$. This result is not surprising as we will see in Section 5. Notice that the constant a is not relevant for practice; it disappears after re-scaling $s \mapsto a\, s$. The trivial case $a = 0$ leads to the non interesting case of the identity operator $\Phi[f, s] = f$ (for all $s > 0$). So Axioms 1–3, 5, 6, 9, 10 impose that the only possible convolution kernels are given by:

$$K_s^{(\alpha)}(\mathbf{x}) = \left[\mathcal{F}^{-1}\left(\omega \mapsto e^{-\|\omega\|^{2\alpha} s}\right)\right](\mathbf{x})$$

Axiom 4 is only satisfied[8] if $\alpha \leq 1$. This easily follows by

$$(-i)^n \int_{\mathbb{R}} x^n \phi(x)\, \mathrm{d}x = \int_{\mathbb{R}} \phi(x) \left(\frac{\partial^n}{\partial \omega^n} e^{-i\omega x} \right)\bigg|_{\omega=0} \mathrm{d}x$$
$$= \sqrt{2\pi} \left(\frac{\partial^n}{\partial \omega^n} \hat{\phi} \right)(0) \quad n \in \mathbb{N}. \tag{11}$$

Take $n = 2$ and notice that $|\omega|^\beta$ is at least twice differentiable and $\hat{\phi}''(0) = 0$ for $\beta > 2$.

Now we have (uniquely) obtained the α-class of scale spaces, we will have a closer look at the non-trivial causality principles. Let's start with Axiom 7b, the strong causality principle. It is shown by Hummel [21] that this principle is equivalent to the following maximum principle:

Definition 1 (Special Cylinder Maximum Principle). Let Ω be a (arbitrary) bounded subset of $\mathbb{R}^d$ and $s_1 > 0$ such that u is continuous on $\bar{\Omega} \times [0, s_1]$, then u attains its maximum or minimum in say $(\mathbf{x}, s) \in \bar{\Omega} \times [0, s_1]$. Either we must have $s = 0$ or $\mathbf{x} \in \partial\Omega$.

Notice that there are quite some maximum principles in analysis, for instance the more famous one for harmonic functions see Theorem 7, therefore we will not speak of *the* maximum principle. Another well known causality constraint in image analysis is Koenderinks principle:

Definition 2 (Koenderink's principle of non-enhancement of local spatial extrema). Let $u(\mathbf{x}, s)$ be a scale space representation then $u_s(\mathbf{x}, s)\, \Delta u(\mathbf{x}, s) > 0$ at spatial extremal points $(\mathbf{x}, s)$, i.e. at points $(\mathbf{x}, s)$ where the spatial gradient $(\nabla_{\mathbf{x}} u)(\mathbf{x}, s) = \mathbf{0}$ and the spatial Hessian $(\nabla_{\mathbf{x}}^2 u)(\mathbf{x}, s)$ is positive or negative definite.

Both Koenderink's principle, the strong causality principle and the special cylinder maximum principle exclude the α scale spaces with $\alpha \neq 1$, while they are all satisfied by the Gaussian case $\alpha = 1$. See [31] for validation of the special cylinder maximum principle in the Gaussian case.

Felsberg [10] has given an example in which he shows that the Koenderink's principle in Poisson scale space ($\alpha = 1/2$) is not satisfied. We have verified (see Fig. 3), that the same holds for the scale spaces $\alpha \in (0.5, 1)$ and that (critical) isophotes in the alpha scale spaces depend continuously on the α parameter. Moreover, the maximum principle follows from the Koenderink's principle: Since Koenderink's principle ensures that local spatial extrema are not enhanced (which follows by a simple Taylor expansion and the fact that $\Delta u = \mathrm{trace}(\nabla^2 u)$), so obviously the extrema will not be in the interior of the cylinder. They can also not lie on top of the cylinder: Suppose there would be say a maximum on top of the cylinder, then from $u \in C^2(\mathbb{R}^+, \mathbb{R}^d)$ it follows that in a small environment within the cylinder around this maximum both $u_s(\mathbf{x}, s)\Delta u(\mathbf{x}, s) > 0$ and $\Delta u(\mathbf{x}, s) < 0$, so in this small environment we have $u_s(\mathbf{x}, s) < 0$, i.e. grey-values decrease towards the the top of the cylinder and therefore the image cannot assume a maximum on the top.

The weak causality principle is satisfied by all α-scale spaces. The fact that Poisson scale space ($\alpha = 1/2$) satisfy the weak causality principle is already shown by Michael Felsberg cf. [10]. A small modification of his proof makes it possible to generalize his result to the α scale spaces. In stead of using the maximum principle (Theorem 7) for harmonic functions in order to exclude the possibility of closed isophotes we use the fact that (see (25))

$$\begin{cases} u_s = -(-\Delta)^\alpha u & \mathbf{x} \in \Omega \\ u|_{\partial\Omega} = c & c \in \mathbb{R} \end{cases}$$

has a unique solution which is given by $u(\mathbf{x}, s) = c$, $\mathbf{x} \in \Omega$ and by Taylor expansion $u(\mathbf{x}, s) = c$ for all $\mathbf{x} \in \mathbb{R}^d$ and $s > 0$, which would imply $f(\mathbf{x}) = c$, contradicting the first Axiom.

4. Strongly Continuous Semi-Groups

In this section we first examine some (non trivial) general theory of strongly continuous semigroups. Afterwards we will focus on the strongly continuous semigroups corresponding to our one parameter class of filters given by (10). The concept of strongly continuous semigroups is important as a neat theoretical

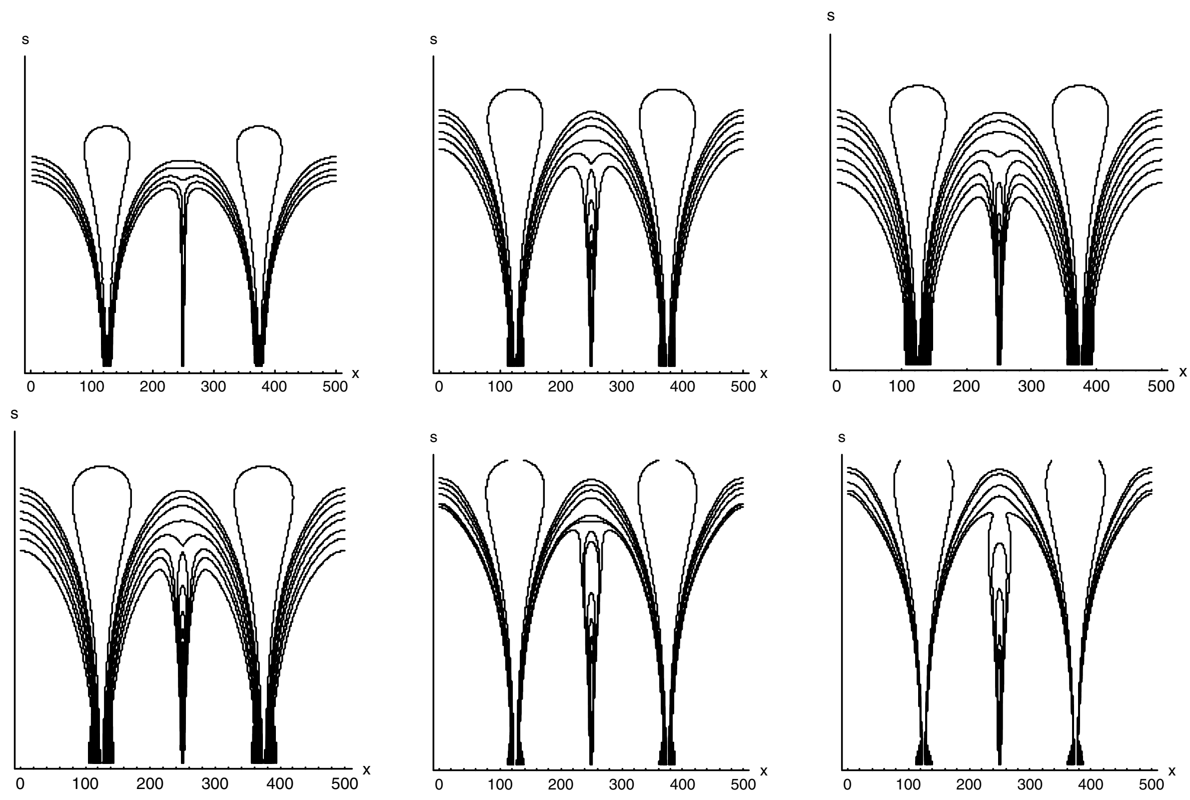


Figure 3. Isophotes of various scale space representations of a signal consisting of 1 small delta spike between two larger delta spikes. Top row: $\alpha = 0.5$ (Poisson scale space), $\alpha = 0.6$, $\alpha = 0.7$, bottom row: $\alpha = 0.8$, $\alpha = 0.9$ and $\alpha = 1$ (Gaussian scale space). The parameter α denotes the fractional power, cf. Eq. (66) . The α scale spaces are sampled according to $s_\alpha = e^{\alpha\tau_n}$, with equidistant τ_n. To this end we notice that both $(s_\alpha)^{\frac{1}{2\alpha}}$ and $\sqrt{s_1} = \sigma$ have dimension [Length], so comparison between scale spaces should always be $(s_\alpha)^{\frac{1}{2\alpha}} \leftrightarrow \lambda\sqrt{s_1} = \lambda\sigma$, where $\lambda > 0$ is some dimensionless constant. Therefore, the stretching of the isophotes as α increases is of no importance. The above figure shows that for each $\alpha \in (0, 1)$ there exist locally concave critical isophotes and the fact that isophotes seem to evolve in a smooth manner as α increases.

approach to Axiom 6 and in particular Axiom 8. The general attitude in image analysis with respect to this is much more sloppy and from a practical and pragmatic point of view this is understandable.

As noticed in Axiom 1 the domain of an original image is assumed to be the whole $\mathbb{R}^d$. In practice images must be extended in some way and thereby external information will be included ! Of course there are other possibilities which we will *not* observe in this article like working on a bounded domain and imposing $\frac{\partial u}{\partial n} = 0$ at its boundary,[9] cf. [5, 8] or working with a torus (periodic extensions). With the eye on all these alternatives (each having their own advantages and disadvantages) we will first observe a more general concept of strongly continuous (semi-)groups, namely continuous representations of Lie-groups G, with identity I, into a Banach space X. For the sake of clarity some proofs of theorems within this section are omitted. They can be found in [4] and [6].

Definition 3. Γ is a bounded continuous representation of G into a Banach space X if

1. $\Gamma(g) \in \mathcal{L}(X)$ and $\sup_{g\in G} \|\Gamma(g)\|_{\mathcal{L}(X)} < \infty$ for all $g \in G$.
2. the mapping $g \mapsto \Gamma(g)$ from G into $\mathcal{L}(X)$ is a homomorphism.
3. $\lim_{g\to I}[\Gamma(g)]x = x$ for all $x \in X$.

Examples:

- The left regular representation L: $X = \mathbb{L}_p(G)$, $(L_g\phi)(h) = \phi(g^{-1}h)$.
  In particular:
  $G = (\mathbb{R}^d, +)$, $X = \mathbb{L}_p(\mathbb{R})$, $[L_\mathbf{t}\phi](\mathbf{x}) = \phi(\mathbf{x} - \mathbf{t})$.

- $G = (SO(d), \cdot)$, $X = \mathbb{L}_2(\mathbb{R}^d)$ $(d \in \mathbb{N})$, $[\mathcal{P}_R\phi](\mathbf{x}) = \phi(R^{-1}\mathbf{x})$

The specific case if $G = (\mathbb{R}^+, +)$ with the extra restriction $s > 0$ leads to following definition.

Definition 4 (Strongly Continuous Semi-group). Let X be a Banach Space, and suppose that to every $s > 0$ is associated an operator $Q_s \in B(X)$, in such way that

- $Q_{s_1+s_2} = Q_{s_1} Q_{s_2}$ for all $s_1, s_2 > 0$
- $\lim_{s\to 0} \|Q_s x - x\| = 0$ for every $x \in X$.

Then $s \mapsto Q_s$ will be called a strongly continuous semigroup.

It can be shown, cf. [6], that a semigroup is strongly continuous if and only if it is weak-weak continuous and $\lim_{s\downarrow 0}\langle f, Q_s x\rangle = \langle f, x\rangle$ for all $x \in X$ and $f \in X'$.

Definition 5. Let G be a Lie-group. A sequence $\{e_n\} \subset \mathbb{L}_1(G)$ is a bounded approximation of unity if

1. $\sup_{n\in\mathbb{N}} \|e_n\|_{\mathbb{L}_1(G)} < \infty$
2. $\lim_{n\to\infty} \int_G e_n \,\mathrm{dg} = 1$
3. The following equality must hold for every neighborhood V of $\mathbf{0}$:

$$\lim_{n\to\infty} \int_{G\setminus V} |e_n| \,\mathrm{dg} = 0 \tag{12}$$

Recall that

$$\int_G f(g)\,\mathrm{dg} = \int_{\mathrm{W}\subset\mathbb{R}^{\mathrm{d}}} \mathrm{f}(\mathrm{g}(\varphi_1, \ldots, \varphi_{\mathrm{d}})\,\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_{\mathrm{d}},$$

where the mapping $(\varphi_1, \ldots, \varphi_d) \subset W \subset \mathbb{R}^d$ onto $g(\varphi_1, \ldots, \varphi_d)$ is a smooth parameterization of G.

Definition 6. Let X be a reflexive Banach space and let Q be a continuous representation of some Lie-group G into X. Let $\psi \in \mathbb{L}_1(G)$, then we define the operator $\mathcal{Q}(\psi) : X \to X$, by

$$[\mathcal{Q}(\psi)]x = \int_G \psi(g) Q_g x \,\mathrm{dg} \tag{13}$$

i.e.

$$\langle f, \mathcal{Q}(\psi)x\rangle = \int_G \psi(g)\langle f, Q_g x\rangle \,\mathrm{dg} \quad \text{for all } \mathrm{f} \in \mathrm{X}'. \tag{14}$$

Note that (14) indeed defines $\mathcal{Q}$, since in a reflexive Banach space $\hat{x}(f) = f(x)$ defines an isomorphism between X'' and X. Further note that $\langle f, Q_g x\rangle$ is uniformly bounded in g ;

$$|\langle f, Q_g x\rangle| \le \|f\|\|Q_g\|\|x\| \le C\|f\|\|x\| \tag{15}$$

for a certain $C > 0$ and $\psi \in \mathbb{L}(G)$, so the integral in the right-hand side of (14) is convergent. Moreover, it follows by (15) that $\mathcal{Q}$ is a bounded operator on X.

If we take $Q = L$ the left regular representation, then we obtain

$$\mathcal{L}(\psi)\phi = \psi * \phi \text{ i.e.}$$
$$[\mathcal{L}(\psi)\phi](h) = \int_G \psi(g)\phi(g^{-1}h)\,\mathrm{dg}$$

Theorem 2. *Let X be a reflexive Banach space. Let Q be a continuous representation of some Lie-Group G into X. Let $\{e_n\}$ be a bounded approximation of unity, then*

$$\lim_{n\to\infty} \mathcal{Q}(e_n)x = x \text{ in } X, \quad \text{for all } x \in X.$$

Proof: Let $x \in X$ and let $\epsilon > 0$. From the third condition of Definition 3 it follows that there exists a neighborhood $V \subset G$ of the unity I in G such that $\|Q_g x - x\| < \frac{\epsilon}{3\sup\|e_n\|_{\mathbb{L}_1(\mathbb{R}^d)}}$ for all $g \in V$. But then we have

$$\begin{aligned}
&\|\mathcal{Q}(e_n)x - x\| \\
&= \left\| \int_G e_n(g) Q_g x \,\mathrm{dg} - \mathrm{x} \right\| \\
&\le \left\| \int_G e_n(g) Q_g x \,\mathrm{dg} - \int_G e_n(g)\, x \,\mathrm{dg} \right\| \\
&\quad + \left\| \left(\int_G e_n(g)\,\mathrm{dg} - 1 \right)\mathrm{x} \right\| \\
&\le \int_V |e_n(g)|\|Q_g x - x\| \,\mathrm{dg} \\
&\quad + \int_{G\setminus V} |e_n(g)|\|Q_g x - x\| \,\mathrm{dg} \\
&\quad + \left| \int_G e_n(g)\,\mathrm{dg} - 1 \right| \|\mathrm{x}\| \\
&\le \frac{\epsilon}{3} + \left(\int_{G\setminus V} |e_n(g)|\,\mathrm{dg} \right) \|\mathrm{x}\| \left(1 + \sup_{\mathrm{g}\in\mathrm{G}} \|\mathrm{Q_g}\| \right) \\
&\quad + \left| \int_G e_n(g)\,\mathrm{dg} - 1 \right| \|\mathrm{x}\|
\end{aligned}$$

Now by the second and third condition of Definition 4 it follows that there exists a $N \in \mathbb{N}$ such that

$$\|Q(e_n)x - x\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for all $n > N$. □

4.1. Strongly Continuity of Poisson and Gaussian Semigroup

Now we focus onto the special case of the Poisson and Gaussian scale spaces:

Take $G = (\mathbb{R}^d, +)$, $X = \mathbb{L}_p(\mathbb{R}^d)$ $(p \geq 1)$, $Q = L$ and let $\{t_n\}$ be a sequence in $\mathbb{R}^+$ such that $t_n \downarrow 0$. Let K_s $(s > 0)$ be a kernel such that

$$\int_{\mathbb{R}^d} K_s(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1,$$
$$K_s \geq 0,$$
$$\lim_{s\downarrow 0} \int_{\mathbb{R}^d \setminus V} |K_s(\mathbf{x})|\,\mathrm{d}\mathbf{x} = 0.$$

For instance the Gaussian kernel $K_s(\mathbf{x}) = G_s(\mathbf{x})$ or the Poisson kernel $K_s(\mathbf{x}) = H_s(\mathbf{x})$. Then $e_n = K_{t_n}$ is bounded approximation of the unity and by the above theorem we obtain:

$$\lim_{n\to\infty} K_{t_n} * \phi = \phi \text{ in } \mathbb{L}_2(\mathbb{R}^d)$$

Since this is valid for all $t_n \to 0$, we obtain

$$\lim_{t\downarrow 0} K_t * \phi = \phi \text{ in } \mathbb{L}_2(\mathbb{R}^d). \tag{16}$$

Applying Fourier Transform with respect to $\mathbf{x}$ to the Dirichlet problem, see (27), respectively Diffusion problem, see (26), defined on the half space $\mathbb{R}^d \times \mathbb{R}^+$ one obtains respectively $\hat{u}(\omega, s) = e^{-|\omega|s}\hat{f}(\omega)$ and $\hat{u}(\omega, s) = e^{-|\omega|^2 s}\hat{f}(\omega)$. Using this together with $\widehat{f * g} = \hat{f}\hat{g}$ and $e^{a+b} = e^a e^b$ leads to (9). To this end we remark that we used that

$$K_{s_1} * (K_{s_2} * f) = (K_{s_1} * K_{s_2}) * f. \tag{17}$$

This equality holds for $f \in \mathbb{L}_2(\mathbb{R}^d)$, since the direct product in the Fourier domain is associative. (If f is a distribution, then things become more complicated). Therefore both the *Gaussian semigroup* $s \mapsto (\phi \mapsto G_s * \phi)$ and the *Poisson semigroup* $s \mapsto (\phi \mapsto H_s * \phi)$ are strongly continuous semigroups on $\mathbb{L}_2(\mathbb{R}^d)$. Finally, we notice that the semigroups corresponding to the α-scale spaces are also strongly continuous which will be shown in Theorem 10.

4.2. Infinitesimal Generators of Strongly Continuous Semigroups and their Resolvents

Given a strongly continuous semigroup Q on a Banach space X, we define the operators $\mathcal{A}_\epsilon$, for $\epsilon > 0$ by

$$A_\epsilon = \frac{Q_\epsilon - I}{\epsilon} \tag{18}$$

Define

$$\mathcal{A}x = \lim_{\epsilon\downarrow 0} \mathcal{A}_\epsilon x \tag{19}$$

for all $x \in \mathcal{D}(\mathcal{A})$, that is , for all $x \in X$ for which the limit (19) exists in the norm topology of X. It is clear that $\mathcal{D}(\mathcal{A})$ is a subspace of X and $\mathcal{A}$ is thus a linear operator in X. This operator, which is essentially $Q'(0)$, is called the *infinitesimal generator* of the semigroup Q.

A bounded operator $\mathcal{A}$ on a Banach space X, which is prescribed on a dense subset has a unique extension on X, i.e. $\mathcal{A}$ is densely defined. For if $x_n \to x$, then $\mathcal{A}x_n \to \mathcal{A}x$. An unbounded operator must be closed to have this property. This means that for all sequences in X, which converge to say $x \in X$ and whose images $\{\mathcal{A}x_n\}$ happen to converge, the limit should be equal to $\mathcal{A}x$. The next theorem shows that an infinitesimal generator of a strongly continuous semi-group, which might be unbounded, is indeed densely defined.

Theorem 3. *Let Q be a strongly continuous semigroup on Banach Space X. Let $\mathcal{A}$ be its infinitesimal generator, then the domain of the infinitesimal generator is dense in X, i.e. $\bar{\mathcal{D}(A)} = X$. Moreover, since $\mathcal{A}$ is a closed operator, it is thereby densely defined.*

The next theorem gives an explicit expression for the resolvent of a strongly continuous semigroup.

Theorem 4. *Consider a semi group Q with infinitesimal generator $\mathcal{A}$. Put*[10]

$$\omega = \lim_{s\to\infty} \frac{\log \|Q_s\|}{s}. \tag{20}$$

Then for $\lambda \in \mathbb{C}$, *with* $\Re(\lambda) > \omega$, *one has* $\lambda \in \rho(A)$ *and*

$$(\lambda I - \mathcal{A})^{-1}x = R(\lambda; \mathcal{A})x = \int_0^\infty e^{-\lambda s} Q_s x \, \mathrm{ds} \quad (21)$$

Remarks:

- Take $X = \mathbb{L}_2(\mathbb{R}^d)$. Let $f \in \mathbb{L}_2(\mathbb{R}^d)$. Let $\mathbf{x} \in \mathbb{R}^d$. Then Eq. (21) states that $[R(\lambda; \mathcal{A})f](\mathbf{x})$ is in fact the Laplace transform of $s \mapsto Q_s f(\mathbf{x})$ evaluated at λ. This is not surprising, since application of Laplace transform onto the evolution equation

$$\begin{cases} \frac{\partial}{\partial s} u = \mathcal{A} u \\ u(\mathbf{x}, 0) = f(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d \end{cases}$$

gives

$$\begin{aligned} \lambda\, \mathcal{L}(u(\mathbf{x}, \cdot))(\lambda) - u(\mathbf{x}, 0) &= \lambda \mathcal{L}(u(\mathbf{x}, \cdot))(\lambda) - f(\mathbf{x}) \\ &= \mathcal{A}\, \mathcal{L}(u(\mathbf{x}, \cdot))(\lambda), \end{aligned}$$

so therefore

$$\begin{aligned} \mathcal{L}(u(\mathbf{x}, \cdot))(\lambda) &= \int_0^\infty u(\mathbf{x}, s) e^{-\lambda s} \, \mathrm{ds} \\ &= [R(\lambda; \mathcal{A})f](\mathbf{x}). \end{aligned}$$

Thereby the resolvent of the generator of the α scale space, $\mathcal{A} = -(-\Delta)^\alpha$, which is studied in detail in Section 6, is a convolution with the Laplace transform of the α-convolution kernel *with respect to s*:

$$[R(\lambda; -(-\Delta)^\alpha)f](\mathbf{x}) = \left[\mathcal{L}\left(s \mapsto K_s^{(\alpha)}\right) * f\right](\mathbf{x}).$$

The Laplace transform of the 1D respectively 2D Gaussian kernel is given by $\frac{e^{-\sqrt{a}x}}{2\sqrt{a}}$ respectively $\frac{K_0(\sqrt{a}r)}{2\pi}$, where K_0 is the zeroth order modified Bessel function of the second kind.

- For *uniformly* continuous semigroups, i.e. semigroups for which $s \to Q_s$ is continuous as a mapping $[0, \infty) \to \mathcal{B}(X)$ we even have

$$R(\lambda; \mathcal{A}) = \int_0^\infty e^{-\lambda s} Q_s \, \mathrm{ds}, \quad \text{for } \Re(\lambda) > \|\mathcal{A}\|. \quad (22)$$

For these semigroups we have $\mathcal{Q}_s = e^{s\mathcal{A}}$, so (22) is a generalization of the well-known formula

$$\mathcal{L}(s \mapsto e^{as})(\lambda) = \int_0^\infty e^{(a-\lambda)s} \, \mathrm{ds} = \frac{1}{\lambda - \mathrm{a}},$$

for all $a \in \mathbb{C} \; : \; \Re(\lambda) > |a|$.

4.3. Holomorphic Semigroups

In Theorem 10 we will show that all α-semigroups and in particular the Poisson and the Gaussian semigroup on $\mathbb{L}_2(\mathbb{R})$ are holomorphic.[11] For these filtering processes this means that as soon as a source image f is filtered with a finite scale $s > 0$ it can be expanded in a Taylor series with respect to $s > 0$.

A holomorphic semigroup on a Banach space X is a semigroup Q, such that Q has a holomorphic extension into a cone $\{\lambda \in \mathbb{C} \; : \; |\arg \lambda| < \arctan \frac{1}{\alpha e}\}$, $\alpha > 0$ in the complex plane, locally given by $Q_\lambda x = \sum_{n=0}^\infty \frac{(\lambda - s)^n}{n!} Q_s^{(n)} x$, $x \in X$.

Yosida [39] p. 255, has shown that holomorphic semigroups are exactly those semigroups which satisfy

$$\begin{aligned} &1. \;\; Q_s x \in \mathcal{D}(\mathcal{A}), \quad \text{for all } s > 0, x \in X \\ &2. \;\; \|s\, Q_s'\| = \|s\, \mathcal{A} Q_s\| \le \alpha \quad \text{for all } s \in (0, 1]. \end{aligned} \quad (23)$$

We will mention some regularity results, that played an important role in the proof of the above statement. For their proofs, see [4].

Lemma 2. *Let Q be a strongly continuous semigroup, with infinitesimal generator $\mathcal{A}$ in a Banach space X. Let $n \in \mathbb{N}$ and $x \in \mathcal{D}(\mathcal{A}^n)$. Then*

1. $Q_s x \in \mathcal{D}(\mathcal{A}^n)$ *for all* $s > 0$, $x \in X$.
2. $Q_s x$ *is n times continuously differentiable with respect to s*
3. $Q_s^{(n)}(x) = \mathcal{A}^n Q_s x = Q_s \mathcal{A}^n x$ $s > 0$, $x \in X$.

Theorem 5. *Let Q be a strongly continuous semigroup, with infinitesimal generator $\mathcal{A}$ in a Banach space X. Suppose $Q_s x \in \mathcal{D}(\mathcal{A})$ for all $x \in X$ and $s > 0$. Then*

$$\begin{cases} Q_s x \in \mathcal{D}(\mathcal{A}^n) \text{ for all } s > 0, x \in X \\ Q_s x \text{ is } n \text{ times differentiable with respect to } s \\ Q_s^{(n)}(x) = \mathcal{A}^n Q_s x = \left(Q'_{\frac{s}{n}}\right)^n x = \left(\mathcal{A} Q_{\frac{s}{n}}\right)^n x, \\ s > 0, x \in X \end{cases} \quad (24)$$

for all $n \in \mathbb{N}$.

5. Evolution Equations Corresponding to α-Scale Spaces

The infinitesimal generator of the α-semigroup given by $s \mapsto (f \mapsto K_s^{(\alpha)} * f)$ corresponding to the α scale space is given by $-(-\Delta)^\alpha$, which will be explained in Section 6 (Theorem 10) in detail. In other words the α scale spaces given by $u(\mathbf{x}, s) = (K_s^{(\alpha)} * f)(\mathbf{x})$ satisfy the pseudo differential evolution system

$$\begin{cases} u_s = -(-\Delta)^\alpha u \\ \lim_{s \downarrow 0} u(\cdot, s) = f(\cdot). \end{cases} \tag{25}$$

A semi-group and in particular corresponding to the α scale spaces is completely determined by (the spectral decomposition of) its generator. It is not difficult to give a *heuristical* impression of how Axioms 1–3, 5, 6 (first part of) 8, 9, 10 lead to the generators $\mathcal{A} = -(-\Delta)^\alpha$:

1. By Axiom 5 and 8 it follows that are indeed strongly semigroups with infinitesimal generator $\mathcal{A}$ which correspond to the possible scale spaces.
2. By rotational invariance, the fact that $\mathcal{F}\mathcal{P}_R = \mathcal{P}_R\mathcal{F}$ and by Plancherels theorem which states that $\mathcal{F}$ is an isometry from $\mathbb{L}_2(\mathbb{R}^d)$ into itself it follows that the corresponding operator in the Fourier domain is functionally dependent on $\|\omega\|^2$.
3. By $[\mathcal{F}(\partial_x f)](\omega) = i\omega\mathcal{F}(\omega)$ this means that the generator must be functionally dependent on Δ: $\mathcal{A} = f(\Delta)$.
4. Since scale space solutions are not allowed to explode as $s \to \infty$ (Axiom 10) we have $f(\Delta) < 0$.
5. By Axiom 3 (dilation invariance) and $\frac{1}{\lambda^2}\Sigma_\lambda\Delta = \Delta\Sigma_\lambda$ it follows that f must be a homogeneous polynomial of one variable, i.e. a monomial $\mathcal{A} = -(-\Delta)^\alpha$ and by the positivity axiom $\alpha < 1$. The cases $\alpha < 0$ are not allowed since their corresponding scale spaces explode and the case $\alpha = 0 \Rightarrow \mathcal{A} = \mathcal{I}$ is not allowed by the average grey-value axiom.

In this section we will mainly focus to the case $\alpha = \frac{1}{2}$, which corresponds to Poisson scale space. But first we will have a short look to the familiar Gaussian case ($\alpha = 1$).

5.1. *The Diffusion Equation*

Definition 7. The Diffusion problem on the half space $\mathbb{R}^d \times \mathbb{R}^+$ is defined by:

$$\begin{array}{ll} [\partial_s - \Delta]u = u_s - \Delta u = 0 & \mathbf{x} \in \mathbb{R}^d, s > 0 \\ \lim_{s \downarrow 0} u(\mathbf{x}, s) = f(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \end{array} \tag{26}$$

If we apply Fourier Transform we obtain the unique solution of this problem, namely $u(\mathbf{x}, s) = \mathcal{F}^{-1}(\omega \mapsto e^{-|\omega|^2 s}\hat{f}(\omega))(\mathbf{x}) = (G_s * f)(\mathbf{x})$. So filtering with Gaussian kernels corresponds to solving the diffusion system (26).

It is well known in the scale space community that filtering with Gaussian kernels satisfies all axioms mentioned in Section 3. See for instance (Sporring-Nielsen-Florack-Johansen [35] Section 4). Therefore we will only mention some details.

In Section 4 it is shown that the mapping from $\mathbb{R}^+ \to B(\mathbb{L}_2(\mathbb{R}))$ given by $s \mapsto (f \mapsto (G_s * f))$ is a *strongly continuous* semi-group. Further on one can show in exactly the same matter as was done in Theorem 6 that $\lim_{s\downarrow 0} |(G_s * f)(\mathbf{x}) - f(\mathbf{x})| = 0$ for all $\mathbf{x} \in \mathbb{R}^d$, whenever f is bounded and continuous on $\mathbb{R}$.

J.Weickert has shown a general theorem from which it follows the axiom on increase of entropy is satisfied in a Gaussian scale space. See Weickert [37] p. 67 Theorem 3. Nevertheless, the result easily follows by substituting $u_s = \Delta u$ in equality (30) and use Greens second identity (i.e. the fact that Δ is self-adjoint).

5.2. *The Poisson Equation*

Definition 8. The Dirichlet problem on the half space $\mathbb{R}^d \times \mathbb{R}^+$ is defined by

$$\begin{array}{ll} \Delta_{\mathbf{x},s} u = \Delta u + u_{ss} = 0 & \mathbf{x} \in \mathbb{R}^d, s > 0 \\ \lim_{s \downarrow 0} u(\mathbf{x}, s) = f(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \end{array} \tag{27}$$

If we apply Fourier Transform we obtain the solution of this problem, namely $\mathcal{F}^{-1}(\omega \mapsto e^{-\|\omega\|s}\hat{f}(\omega)) = \mathcal{F}^{-1}(\omega \mapsto e^{-\|\omega\|s}) * f = H_s * f$. See Section 5.2.3 for an alternative derivation, using Greens function.

5.2.1. *Explicit Verification of All Axioms.* In this paragraph we will show that the mapping $\Phi : \mathbb{L}_2(\mathbb{R}^d) \times \mathbb{R}^+ \to \mathbb{L}_2(\mathbb{R}^d)$ given by $\Phi[f, s] = H_s * f$ satisfies *all* axioms. It is trivial that Axiom 1–3 are satisfied. Axiom 4 is satisfied since the kernel H_s is positive.

Let $s > 0$ be fixed. Note that $f \mapsto H_s * f$ is a bounded operator, since the kernel is an element of $\mathbb{L}_1(\mathbb{R}^d)$ (see Theorem 1), and thereby Axiom 5a is satisfied. We even have $H_s \in \mathbb{L}_2(\mathbb{R}^d)$ so it is also a bounded operator from $\mathbb{L}_2(\mathbb{R}^d)$ into $\mathbb{L}_\infty(\mathbb{R}^d)$, since by Cauchy-Schwarz

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |(H_s * f)(\mathbf{x})| \leq \|H_s\|_{\mathbb{L}_2(\mathbb{R}^d)} \|f\|_{\mathbb{L}_2(\mathbb{R}^d)}. \tag{28}$$

So Axiom 5b is also satisfied. Axiom 6 and the first part of Axiom 8 are already proven in Section 4. The rest will be proven in Theorem 6. We have already noticed in Section 3 that all α scale spaces and in particular $\alpha = 1/2$ obey the weak causality principle, Axiom 7a. Although harmonic functions satisfy the mean value principle[12] and main maximum principle functions, they do not satisfy the special cylinder maximum principle, cf. Definition 1, since extrema can lie on top of the cylinder. This coincides with the fact that the strong causality axiom and Koenderink's principle are not satisfied when $\alpha = 1/2$. An easy example of a harmonic function which doesn't satisfy Koenderink's principle is given by[13] $h(x, y, s) = \cos(s\sqrt{2}) \cosh x \cosh y$.

Axiom 9 is obviously satisfied since the Kernel only depends on $\|\mathbf{x}\|$ and for the verification of Axiom 10 we only need to show that the $\mathbb{L}_1$-norm of the Poisson kernel equals 1:

$$\begin{aligned}\|H_s\|_{\mathbb{L}_1(\mathbb{R}^d)} &= \frac{2}{\sigma_{d+1}} \int_{\mathbb{R}^d} \frac{s}{(s^2 + \|\mathbf{x}\|^2)^{\frac{d}{2}}} \, d\mathbf{x} \\ &= \frac{2\sigma_d}{\sigma_{d+1}} \int_0^\infty \frac{s\, r^{d-1}}{(s^2 + r^2)^{\frac{d}{2}}} \, dr \\ &= \frac{2\sigma_d}{\sigma_{d+1}} \frac{\sqrt{\pi}\,\Gamma(d/2)}{2\Gamma((d+1)/2)} = 1.\end{aligned} \tag{29}$$

For the verification of the entropy axiom, Axiom 11, see Theorem 8.

Theorem 6. *Let $f \in \mathbb{L}_2(\mathbb{R}^d)$ and suppose f is continuous on $\mathbb{R}^d$ and bounded on $\mathbb{R}^d$ i.e. $\sup_{\boldsymbol{x}\in\mathbb{R}^d} |f(\boldsymbol{x})| = M < \infty$, then*

$$\lim_{s\downarrow 0} |(H_s * f)(\boldsymbol{x}) - f(\boldsymbol{x})| = 0 \quad \textit{for all } \boldsymbol{x} \in \mathbb{R}^d.$$

Proof: Let $\mathbf{x} \in \mathbb{R}^n$ and let $\epsilon > 0$.

Since f is continue, there exists a $\delta > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{y})| < \frac{\epsilon}{2} \quad \text{for all } \mathbf{y} \in B_{\mathbf{x},\delta}.$$

As a result

$$\begin{aligned}&|(H_s * f)(\mathbf{x}) - f(\mathbf{x})| \\ &\quad = \frac{2s}{\sigma_{d+1}} \left| \int_{\mathbb{R}^2} \frac{f(\mathbf{y}) - f(\mathbf{x})}{(\|\mathbf{x}-\mathbf{y}\|^2 + s^2)^{\frac{d+1}{2}}} \, d\mathbf{y} \right| \\ &\quad \leq \frac{2s}{\sigma_{d+1}} \int_{B_{\mathbf{x},\delta}} + \int_{\mathbb{R}^2 \setminus B_{\mathbf{x},\delta}} \frac{|f(\mathbf{y}) - f(\mathbf{x})|}{(\|\mathbf{x}-\mathbf{y}\|^2 + s^2)^{\frac{d+1}{2}}} \, d\mathbf{y} \\ &\quad \leq \frac{\epsilon}{2} + \frac{4M\, s\, \sigma_d}{\sigma_{d+1}} \int_\delta^\infty r^{-2} \, dr\end{aligned}$$

So there exists an $S > 0$ small enough such that

$$|(H_s * f)(\mathbf{x}) - f(\mathbf{x})| < \epsilon \quad \text{for all } s < S.$$

□

In particular we have

$$\langle \delta_{\mathbf{x}}, f \rangle = f(\mathbf{x}) = \lim_{s\downarrow 0} (H_s * f)(\mathbf{x})$$

for all $f \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{x} \in \mathbb{R}^d$.

So, if we denote the mapping $f \mapsto (H_s * f)(\mathbf{x})$ by $Q_s^{\mathbf{x}}$ then

$$D^\alpha Q_s^{\mathbf{x}} \to D^\alpha \delta_{\mathbf{x}} \text{ in } \mathcal{D}'(\mathbb{R}^d) \text{ for every multi-index } \alpha.$$

Theorem 7 (*The main maximum principle for harmonic functions*). *Let Ω be a connected open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Let f be a harmonic function on Ω. If f attains its maximum at $\boldsymbol{a} \in \Omega$, then f is a constant.*

Proof: Let M denote the set of maximum points of f in Ω. Then M is closed in Ω since f is continuous. The set M is also open, because of the mean value theorem for harmonic functions.[14] But since Ω is connected, the only open and closed subsets of Ω are Ω and the empty set. □

If Ω is bounded and if f is continuous on $\bar{\Omega}$, then f attains a maximum and minimum on $\bar{\Omega}$. From Theorem 7 it follows that this extreme points lie on $\partial\Omega$.

Theorem 8. *Let u be the Poisson scale space of a positive image, i.e. $u(\boldsymbol{x}, s) = (H_s * f)(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$, $s > 0$, for a certain $f \in \mathbb{L}_2(\mathbb{R}^d)$ such that $0 < f < 1$ almost every where. Let $\mathcal{E}(u) : \mathbb{R}^+ \to \mathbb{R}$ be defined by (7). Then $s \mapsto [\mathcal{E}(u)](s)$ is monotonically increasing. Moreover, $\frac{\partial}{\partial s}\mathcal{E}_s(u) \to 0$ for $s \to \infty$.*

Proof: First, we will show that the second order partial derivative to s is negative. We use $u_{ss} = -\Delta u$ and at the end Greens first identity (partial integration).

$$\begin{aligned}
\frac{\partial^2[\mathcal{E}(u)](s)}{\partial s^2} &= -\int_{\mathbb{R}^d} \frac{\partial^2}{\partial s^2}(u(\mathbf{x}, s) \ln u(\mathbf{x}, s))\, d\mathbf{x} \\
&= -\int_{\mathbb{R}^d} \frac{(u_s(\mathbf{x}, s))^2}{u(\mathbf{x}, s)}\, d\mathbf{x} \\
&\quad - \int_{\Omega} (\ln u(\mathbf{x}, s) + 1) u_{ss}(\mathbf{x}, s)\, d\mathbf{x} \\
&< \int_{\mathbb{R}^d} (\ln u(\mathbf{x}, s) + 1)\Delta u(\mathbf{x}, s)\, d\mathbf{x} \\
&= \int_{\mathbb{R}^d} \ln u(\mathbf{x}, s)\, \Delta u(\mathbf{x}, s)\, d\mathbf{x} \\
&= -\int_{\mathbb{R}^d} \frac{\|\nabla u(\mathbf{x}, s)\|^2}{u(\mathbf{x}, s)}\, d\mathbf{x} \leq 0
\end{aligned}$$

Notice with respect to the first inequality that the positivity axiom and $0 < f < 1$ imply that $0 < u(\cdot, s) < 1$ for all $s > 0$. Next, we note that $\lim_{s\to\infty} \frac{\partial}{\partial s}[\mathcal{E}(u)](s) = 0$. This follows from

$$\frac{\partial}{\partial s}[\mathcal{E}(u)](s) = -\int_{\mathbb{R}^d} (\ln u + 1)\frac{\partial u}{\partial s}\, d\mathbf{x} \tag{30}$$

and the fact that $\lim_{s\to\infty} \frac{\partial}{\partial s} u(\mathbf{x}, s) = 0$. Finally, we will show that $s \mapsto \frac{\partial[\mathcal{E}(u)](s)}{\partial s}$, is continuous on $(0, \infty)$:

Let $t > 0$. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$, such that $t_n \to t \ (n \to \infty)$.

By Lebesgue's dominated convergence principle and the fact that u is continuously differentiable with respect to s.

$$\begin{aligned}
\lim_{t_n\to t}[\mathcal{E}(u)](t_n) &= \lim_{t_n\to t} \int_{\Omega} (1 + \ln u(\mathbf{x}, t_n)) u_s(\mathbf{x}, t_n)\, d\mathbf{x} \\
&= \int_{\Omega} \lim_{t_n\to t} (1 + \ln u(\mathbf{x}, t_n)) u_s(\mathbf{x}, t_n)\, d\mathbf{x} \\
&= [\mathcal{E}(u)](t)
\end{aligned}$$

As a result we have $\frac{\partial[\mathcal{E}(u)](s)}{\partial s} > 0$ for all $s > 0$. □

5.2.2. The Infinitesimal Generator of the Poisson Semigroup. First we will examine the case $d = 1$ and later we generalize to the case $d > 1$.

By applying Fourier Transform with respect to $\mathbf{x}$ onto the Dirichlet problem one obtains the ordinary differential equation

$$\begin{cases} \dfrac{\partial^2}{\partial s^2}\hat{u}(\omega, s) = \omega^2\, \hat{u}(\omega, s) \\ \hat{u}(\omega, 0) \quad\quad = \hat{f}(\omega). \end{cases}$$

Normally, one would find $\hat{u}(\omega, s) = Ae^{-|\omega|s}\hat{f}(\omega) + Be^{|\omega|s}\hat{f}(\omega)$, but by Plancherel's theorem and the fact that $Q_s \in B(\mathbb{L}_2(\mathbb{R}^1))$, for all $s > 0$, it follows that $B = 0$. So, in the Fourier domain the infinitesimal generator becomes $-|\omega|$. Since operator $\frac{\partial^2}{\partial x^2}$ is negative definite and self-adjoint (Δ in the general case $d \in \mathbb{N}$) we have that the infinitesimal generator of Q equals $-\sqrt{-\frac{\partial^2}{\partial x^2}}$. For more information on fractional powers (see Section 6).

To this end we remark that

$$\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial s} - \sqrt{-\frac{\partial^2}{\partial x^2}}\right)\left(\frac{\partial}{\partial s} + \sqrt{-\frac{\partial^2}{\partial x^2}}\right).$$

Operator $\frac{\partial^2}{\partial x^2}$ is a continuous operator from $\mathbb{H}_s(\mathbb{R})$ $(s > 0)$ into $\mathbb{H}_{s-2}(\mathbb{R})$ and $-\sqrt{-\frac{\partial^2}{\partial x^2}}$ is a continuous operator from $\mathbb{H}_s(\mathbb{R})$ $(s > 0)$ into $\mathbb{H}_{s-1}(\mathbb{R})$. The domain of an infinitesimal operator is always dense in X (see Theorem 3). In our case, we have $X = \mathbb{L}_2(\mathbb{R}) = \mathbb{H}_0(\mathbb{R})$. From Hille [20] (Theorem 21.4.2, p. 576) it follows that

$$\mathcal{D}\left(\sqrt{-\frac{\partial^2}{\partial x^2}}\right) = \{f \in S' \,:\, -|\omega|\hat{f}(\omega)$$

is the Fourier Transform of an element in $\mathbb{L}_2(\mathbb{R})\}$.

Since Fourier Transformation is a unitary operation on $\mathbb{L}_2(\mathbb{R})$ and since $\mathbb{H}_1(\mathbb{R}) \subset \mathbb{L}_2(\mathbb{R})$ we thus obtain that

$$\mathcal{D}\left(\sqrt{-\frac{\partial^2}{\partial x^2}}\right) = \mathbb{H}_1(\mathbb{R}),$$

which is indeed dense in $\mathbb{L}_2(\mathbb{R})$. Next we will show some properties of the infinitesimal generator of Q.

Theorem 9. *The infinitesimal generator* $\mathcal{A} = -\sqrt{-\frac{\partial^2}{\partial x^2}}$ *of the Poisson semi-group* Q *on* $\mathbb{L}_2(\mathbb{R})$ *given by*

$$[Q_s f](x) = (H_s * f)(x) \quad x \in \mathbb{R},\, f \in \mathbb{L}$$

is symmetric, negative definite and satisfies

$$\mathcal{A}f = -Hf' \text{ for all } f \in \mathbb{H}_1(\mathbb{R}), \tag{31}$$

with $H : \mathbb{L}_2(\mathbb{R}) \to \mathbb{L}_2(\mathbb{R})$ *the Hilbert Transform, which can be given by an integral in principal value sense*:

$$(Hf)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,\mathrm{dt}, \tag{32}$$

existing for almost every x.

Proof: First we will show that the Hilbert transform is properly defined by (32):

Let $f \in \mathbb{L}_2(\mathbb{R})$. Define $g_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}$ by

$$g_j(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{1}{x} & \text{if } \frac{1}{j} < |x| < j \\ 0 & \text{else} \end{cases}$$

Using contour integration in the complex plane (Jordan's lemma) one easily finds the pointwise limit

$$\begin{aligned} \lim_{j\to\infty} \hat{g}_j(\omega)\hat{f}(\omega) &= \sqrt{\frac{1}{2\pi}}\sqrt{\frac{2}{\pi}}\hat{f}(\omega)\int_{-\infty,PV}^{\infty} \frac{e^{-i\omega x}}{x}\,\mathrm{dx} \\ &= -\frac{2i}{\pi}\hat{f}(\omega)\lim_{j\to\infty}\int_{1/j}^{j}\frac{\sin(\omega x)}{x}\,\mathrm{dx} \\ &= -i\,\mathrm{sgn}(\omega)\hat{f}(\omega)(\omega \in \mathbb{R}) \end{aligned} \tag{33}$$

and obviously $(\omega \to -i\,\mathrm{sgn}(\omega)f(\omega)) \in \mathbb{L}_2(\mathbb{R})$. Therefore by Lebesgue's dominated convergence principle, $(\omega \to \hat{g}_j(\omega)\hat{f}(\omega))$ converges to $(\omega \to -i\,\mathrm{sgn}(\omega)\hat{f}(\omega))$ in $\mathbb{L}_2$ sense. So, by Plancherel we conclude that $f * g_j$ converges in $\mathbb{L}_2(\mathbb{R})$. This limit is called the Hilbert transform of f and is given by (32).

We have for $f \in \mathcal{D}(\mathcal{A}) = \mathbb{H}_1(\mathbb{R})$

$$\begin{aligned} \mathcal{F}(Hf')(\omega) &= -\lim_{j\to\infty}\mathcal{F}(f' * g_j)(\omega) \\ &= -i\omega\Big(\lim_{j\to\infty}\hat{g}_j(\omega)\Big)\hat{f}(\omega) \\ &= i\omega\, i\,\mathrm{sgn}(w)\hat{f}(\omega) = -|\omega|\hat{f}(\omega) \\ &= \mathcal{F}(\mathcal{A}f)(\omega). \end{aligned} \tag{34}$$

Note that $f' \in \mathbb{L}_2(\mathbb{R})$ for all $f \in \mathbb{H}_1(\mathbb{R})$.

By Plancherel's theorem we now conclude

$$\mathcal{A}f = -Hf' \text{for all } f \in \mathbb{H}_1(\mathbb{R}).$$

Next, we will use this equality in order to show that $\mathcal{A}^* = \mathcal{A}$ and $\mathcal{A} < 0$. As this can only be the case if

$$\begin{cases} 1.\ (Hf', g) = (f, Hg') \\ 2.\ (Hf', f) > 0 \quad \text{for all } f, g \in \mathbb{H}_1(\mathbb{R}). \end{cases}$$

From (33) it follows that

$$\widehat{Hf}(\omega) = -i\,\mathrm{sgn}(\omega)\hat{f}(\omega) \text{ for every } f \in \mathbb{L}_2(\mathbb{R}) \tag{35}$$

Using (35), it is easy to Proof 1 and 2:

1.

$$\begin{aligned} (Hf', g) &= (\widehat{Hf'}, \hat{g}) = (i\,\omega(-i)\,\mathrm{sgn}(w)\,\hat{f}, \hat{g}) \\ &= (\hat{f}, |\omega|\hat{g}) \\ &= (\hat{f}, \widehat{Hg'}) \\ &= (f, Hg') \end{aligned}$$

2.

$$\begin{aligned} (Hf', f) &= (\hat{Hf'}, \hat{f}) = |\omega|\,(\hat{f}, \hat{f}) \\ &= (\hat{f}, |\omega|\hat{f}) > 0 \end{aligned}$$

□

Remarks:

- One can show similarly to the 1D case above, that the Poisson scale space generator in the d-dimensional case is given by

$$-\sqrt{-\Delta} = -\mathbf{R}\cdot\nabla = -\nabla\cdot\mathbf{R} = -\sum_{j=1}^{d} R_j\,\partial_j, \tag{36}$$

where ∂_j is short notation for $\frac{\partial}{\partial x_j}$ and $\mathbf{R} = \sum_{j=1}^{d}\mathbf{e}_j R_j$ denotes the Riesz Transform, which is given by the *principal value* integral

$$R_j f(\mathbf{x}) = \frac{2}{\omega_{d+1}}\int_{\mathbb{R}^d}\frac{x_j - y_j}{\|\mathbf{x}-\mathbf{y}\|^{d+1}} f(\mathbf{y})\,\mathbf{dy} \tag{37}$$

Notice that if $d = 1$ the Riesz Transform equals the Hilbert Transform. Notice that the Gaussian equivalent of (36) is given by $\Delta = \nabla\cdot\nabla$. Consequently, the $(d+1)D$ vector scale space consisting of Gaussian scale space and its first order spatial derivatives corresponds to the Poisson scale space and its Riesz transform components, which is first put in the context of image analysis by Felsberg [9] and which will be further examined in Section 5.2.4:

$$\mathbf{e}_{d+1}(G_s * f)(\mathbf{x}) + \sum_{j=1}^{d} \mathbf{e}_j(\partial_j G_s * f)(\mathbf{x}) \leftrightarrow \mathbf{e}_{d+1}(H_s * f)(\mathbf{x}) + \sum_{j=1}^{d} \mathbf{e}_j(R_j H_s * f)(\mathbf{x}), \quad (38)$$

with respect to image analysis this means that analogue to the fact that $-\nabla(G_s * f)(\mathbf{x})$ equals the *grey-value flow* in a Gaussian scale space $\mathbf{R}(H_s * f)(\mathbf{x})$ describes the *grey-value flow* in a Poisson scale space. Since by Gauss' divergence theorem we have for all $\Omega' \subset \Omega$:

$$\frac{\partial}{\partial s}\int_{\Omega'} u_\alpha(\mathbf{x}, s)\,d\mathbf{x} = \begin{cases} \int_{\partial\Omega'} \nabla u \cdot \mathbf{n}\, d\sigma & \alpha = 1 \\ -\int_{\partial\Omega'} \mathbf{R}u \cdot \mathbf{n}\, d\sigma & \alpha = 1/2 \end{cases}. \quad (39)$$

- In the general d dimensional case we the Laplace operator (with respect to both s and $\mathbf{x}$) can be factorized in an analogue matter:

$$u_{ss} + \Delta u = (\partial_s - \sqrt{-\Delta})(\partial_s + \sqrt{-\Delta})u = 0$$

and since the nill-space of the linear operator in the first factor of the factorization is zero, this equation is equivalent to

$$(\partial_s + \sqrt{-\Delta})u = 0 \Leftrightarrow u_s = -\sqrt{-\Delta}\, u,$$

which indeed corresponds to the pseudo differential equation in (25) when $\alpha = 1/2$.
- Both Property 1 and 2 can also be shown by applying partial integration onto the principal value integral representation for the Hilbert Transform given by (32).
- From (35), it follows that H is an $\mathbb{L}_2(\mathbb{R})$-isometry, of period 4. (Fourier Transform also has this property) Since,

$$\begin{cases} H^2 f = -f \\ \|Hf\|_{\mathbb{L}_2(\mathbb{R})} = \|f\|_{\mathbb{L}_2(\mathbb{R})} \text{ for all } f \in \mathbb{L}_2(\mathbb{R}) \end{cases} \quad (40)$$

Another consequence of the above together with theorem is that

$$\partial^{2k} = (-1)^k \mathcal{A}^{2k}, k \in \mathbb{N} \quad (41)$$

By the second equality of (40) it follows that $\|H\| = 1$ and therefore by (31) it follows that $\|\mathcal{A}\| = 1$, i.e. the according semigroup Q is a contraction semigroup.
- The operators $\mathcal{A} = -\sqrt{-\frac{\partial^2}{\partial x^2}}$, $\partial = \frac{\partial}{\partial x}$ and $\partial^2 = \frac{\partial^2}{\partial x^2}$ respectively the infinitesimal generators of the Poisson, translation and Gaussian semigroup, have all the same smooth elements since

$$\mathcal{D}(\mathcal{A}^\infty) = \bigcap_{n=1}^{\infty} \mathcal{D}(\mathcal{A}^n) = \bigcap_{n=1}^{\infty} \mathbb{H}_n = \bigcap_{n=1}^{\infty} \mathbb{H}_{2n} = \mathbb{H}_\infty$$

and the same analytic elements since the Hilbert transform, which equals the (operator) product of $\mathcal{A}^{-1}\partial$ is unitary on $\mathbb{L}_2(\mathbb{R})$. For a definition of these items (see Appendix). In case we regard the Poisson semigroup on $\mathbb{L}_\infty(\mathbb{R})$ equipped with the sup-norm $\|f\|_{\mathbb{L}_\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |f(x)|$ then these operators still have the same smooth and analytic elements as we will next show, but operators $\mathcal{A}$ is no longer linearly[15] bounded by ∂. Since the Poisson semigroup is a contraction semigroup, we have by equality (21) that the (operator) norm of $R(\epsilon, \mathcal{A}) = (\epsilon I - \mathcal{A})^{-1}$ is at most 1. So $\|(I - \epsilon\mathcal{A})f\| \geq \|f\|$ for all $\epsilon > 0$ and $f \in \mathcal{D}(\mathcal{A}) = \mathbb{H}_1$. Therefore

$$\begin{aligned} \epsilon\|\mathcal{A}f\|_{\mathbb{L}_\infty(\mathbb{R})} &\leq \|(I + \epsilon\mathcal{A})f\|_{\mathbb{L}_\infty(\mathbb{R})} + \|f\|_{\mathbb{L}_\infty(\mathbb{R})} \\ &\leq \|(I - \epsilon^2\mathcal{A}^2)f\|_{\mathbb{L}_\infty(\mathbb{R})} + \|f\|_{\mathbb{L}_\infty(\mathbb{R})} \\ &\leq \epsilon^2\|\mathcal{A}^2 f\|_{\mathbb{L}_\infty(\mathbb{R})} + 2\|f\|_{\mathbb{L}_\infty(\mathbb{R})} \end{aligned}$$

Obviously, $\partial : \mathbb{H}_1 \to \mathbb{L}_2$, satisfies $\|\partial\| \leq 1$ and therefore we can apply the same reasoning on ∂ and obtain a similar estimate. By taking $\mathcal{A}^{2m} f$ respectively $\partial^{2m} f$ in stead of f and $\epsilon = 1$ in these estimates and using (41) we obtain

$$\|\partial^{2m} f\|_{\mathbb{L}_\infty(\mathbb{R})} = \|\partial^{2m} f\|_{\mathbb{L}_\infty(\mathbb{R})}$$

for $f \in \mathbb{H}_{2m}$ and

$$\|\partial^{2m+1} f\|_{\mathbb{L}_\infty(\mathbb{R})} \leq \|\mathcal{A}^{2m+2} f\|_{\mathbb{L}_\infty(\mathbb{R})} + 2\|\mathcal{A}^{2m} f\|_{\mathbb{L}_\infty(\mathbb{R})}$$

$$\|\mathcal{A}^{2m+1} f\|_{\mathbb{L}_\infty(\mathbb{R})} \leq \|\partial^{2m+2} f\|_{\mathbb{L}_\infty(\mathbb{R})} + 2\|\partial^{2m} f\|_{\mathbb{L}_\infty(\mathbb{R})}$$

for $f \in \mathbb{H}_{2m+2}$.

5.2.3. Greens Function on the Half Space $s > 0$. Another way to obtain the solution of the Laplace problem is by using the fundamental solution $S : (\mathbb{R}^d \times \mathbb{R}^+) \times (\mathbb{R}^d \times \mathbb{R}^+) \to \mathbb{R}$ given by

$$\begin{cases} S(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \dfrac{1}{(d-1)\,\sigma_{d+1}} \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|^{1-d} & \text{for } d \geq 2, \\ S(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \dfrac{1}{2\pi} \log \dfrac{1}{\|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|} & \text{for } d = 1, \end{cases} \tag{42}$$

for $\hat{\mathbf{x}} = (\mathbf{x}, s) \neq (\mathbf{y}, t) = \hat{\mathbf{y}}$. This pointwise notation of the fundamental solution might be deceptive, since in strict sense S is a distribution in $\mathcal{D}'(\mathbb{R}^n)$ with non-compact support. However, since Δ is an elliptic operator with constant coefficients the fundamental solution can be regarded as an infinitely differentiable function outside the origin. See, Rudin [33] Theorem 8.12, p. 201.

Define $\hat{\mathbf{y}}* = (\mathbf{y}, -t)$. This point is the result of mirroring $\hat{\mathbf{y}}$ in the plane $s = 0$. Then one can easily verify that Greens function on a half plane is given by[16]

$$G(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = S(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - S(\hat{\mathbf{x}}, \hat{\mathbf{y}}*) \tag{43}$$

Recall Greens second identity on a bounded region Ω with boundary $\partial\Omega$ and outward normal $\mathbf{n}$.

$$\int_\Omega u\Delta v - v\Delta u \,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{n}} - \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{n}} \,\mathrm{d}\sigma$$

Take[17] $v = G$ and

$$\Omega = (\mathbb{R}^d \times \mathbb{R}^+) \cap (B_{\mathbf{0},R} \setminus B_{\mathbf{y},\delta}),$$

with $\delta > 0$ sufficiently small and $R > 0$ sufficiently large. Then one can verify that $\int_{\partial B_{\mathbf{y},\delta}, s>0} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})\, d\sigma \to u(\mathbf{x}, s)$ and $\int_{\partial B_{\mathbf{0},R}} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})\, d\sigma \to 0$ if respectively $\delta \downarrow 0$ and $R \to \infty$. Moreover, Greens function G is harmonic on Ω. As a result we obtain

$$u(\mathbf{x}, s) = \int_{\mathbb{R}^d} f(\mathbf{y}) \left.\frac{\partial G(\hat{\mathbf{x}}, \hat{\mathbf{y}})}{\partial t}\right|_{t=0} \mathrm{d}\sigma_{\mathbf{y}}. \tag{44}$$

Now by Eq. (42) we find

$$\begin{aligned} \left.\frac{\partial G(\hat{\mathbf{x}}, \hat{\mathbf{y}})}{\partial t}\right|_{t=0} &= \left.\frac{\partial S(\hat{\mathbf{x}}, \hat{\mathbf{y}})}{\partial t}\right|_{t=0} - \left.\frac{\partial S(\hat{\mathbf{x}}, \hat{\mathbf{y}}*)}{\partial t}\right|_{t=0} \\ &= 2\left.\frac{\partial S(\hat{\mathbf{x}}, \hat{\mathbf{y}})}{\partial t}\right|_{t=0} = \frac{2}{\sigma_{d+1}} \frac{s}{(s^2 + \|\mathbf{x} - \mathbf{y}\|^2)^{\frac{d+1}{2}}} \\ &= H_s(\mathbf{x} - \mathbf{y}) \end{aligned}$$

and substituting this result into Eq. (44) we indeed find:

$$u(\mathbf{x}, s) = (H_s * f)(\mathbf{x}).$$

5.2.4. Clifford Analytic Extension of Poisson Scale Space. In this subsection we give theoretic background to a highly interesting new approach to scale space theory which is first introduced by Felsberg and Sommer [9].

In case of 1D-signals ($d = 1$) it is possible to extend the Poisson scale space to an analytic scale space $\tilde{u}(x + is) = u_A(x, s) = u(x, s) + iv(x, s)$, simply by adding i times the harmonic conjugate v which is determined (up to a constant) by the Cauchy-Riemann equations $u_x = v_s$ and $u_s = -v_x$. The harmonic conjugate is given by $v(x, s) = (Q_s * f)(x, s)$, where Q_s denotes the conjugate Poisson kernel which is given by the Hilbert transform of the Poisson kernel:

$$Q_s(\mathbf{x}) = (HH_s)(x) = \frac{1}{\pi} \frac{x}{s^2 + x^2}.$$

This follows directly by Cauchy's integral formula for analytic functions:

$$\tilde{u}(z) = \frac{1}{2\pi i} \oint_C \frac{\tilde{u}(w)}{w - z} \, dw \quad z = x + is, \tag{45}$$

where C is any positively oriented simple curve around z, since

$$\begin{aligned} H_s(x) &= \Re\left(\frac{1}{(2\pi i)(z)}\right) \\ Q_s(x) &= \Im\left(\frac{1}{(2\pi i)(z)}\right). \end{aligned}$$

In particular by taking $C = C_0 \cup C_R \cup C_\delta$, with $C_0 = [-R, R]$ and $C_R = \{z \in \mathbb{C}_+ \mid |z| = R\}$, $C_\delta = \{z \in \mathbb{C}_+ \mid |z| = \delta\}$ in (45) and letting $\delta \to 0$, $R \to \infty$ we obtain the Cauchy operator $C : \mathbb{L}_2(\mathbb{R}) \to H^2(\mathbb{C}_+)$

which is given by

$$\begin{aligned}(Cf)(x,s) &= \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z}\,\mathrm{dt}\\ &= \frac{1}{2}((H_s * f)(x) + i\,(Q_s * f)(x)),\\ z &= x+is \in \mathbb{C}_+,\end{aligned}$$

where the space $H^2(\mathbb{C}_+)$ consists of all analytic functions F on $\mathbb{C}_+$ such that

$$\sup_{t>0}\int_{-\infty}^{\infty} |F(x+it)|^2\,\mathrm{dx} < \infty.$$

Any signal can be split uniquely and orthogonally into an analytic and a non-analytic part:

$$\begin{aligned}\mathbb{L}_2(\mathbb{R}) &= H^2(\partial\mathbb{C}_+) \oplus (H^2(\partial\mathbb{C}_+))^{\perp},\\ f &= f_{AN} + f_{NAN} = \frac{f+iHf}{2} + \frac{f-iHf}{2}.\end{aligned}$$

where the subspace of analytic signals is given by

$$H^2(\partial\mathbb{C}_+) = \{f \in \mathbb{L}_2(\mathbb{R}) \mid \mathrm{supp}(f) \subset [0,\infty)\}.$$

To this end we recall (35) so indeed $f_{AN}(\omega) = 0$ for $\omega < 0$. Further we notice[18]

$$\begin{aligned}&Cf = C(f_{AN}) + C(f_{NAN}) = C(f_{AN}) + 0\\ &\lim_{s\downarrow 0} Cf(\cdot,s) = f_{AN}.\end{aligned}$$

In practice f is real valued, so then $f = 2\Re(f_{AN})$ and consequently

$$u(x,s) = \Re\tilde{u}(x,s) = 2\Re(Cf_{AN})(x,s) = (H_s * f)(x).$$

Remarks

- Physically, the Poisson scale space should be regarded as a potential problem rather than a heat problem. The isophotes within the Poisson scale space correspond to equi-potential curves and the isophotes within the conjugate Poisson scale space correspond to the flow-lines. By the Cauchy-Riemann equations these lines intersect each other orthogonal through each point (x,s):

$$(\partial_x,\partial_s)u \cdot (\partial_x,\partial_s)v = u_x v_s + u_s v_x = 0.$$

  For instance the isophotes of the Poisson kernel $K^{(1/2)}$ are the semi-circles $x^2 + (s-a)^2 = a^2, a, s > 0$, $x \in \mathbb{R}$ which intersect the flow lines $(x+a)^2 + s^2 = a^2$, $a, x \in \mathbb{R}, s > 0$ orthogonal. It might be tempting to regard f as charge density distribution, but this is not right: f is the potential at the boundary, due to some charge-distribution in the plane $s < 0$. By writing $u = f + \mathcal{D}(\Delta f)$, where $\mathcal{D}$ denotes the Dirichlet operator (i.e. $\Delta_{\mathbf{x},s}\mathcal{D}f = -f$ and $\mathcal{D}(f)(\mathbf{x},0) = 0$) it is possible to regard Δf as a charge density function (independent of $s > 0$).
- The 2D Laplace operator can be split into two different ways:

$$\begin{aligned}\Delta_2 &= (\partial_s + i\partial_x)(\partial_s - i\partial_x) = 4\partial_{\bar{z}}\partial_z\\ \Delta_2 &= (\partial_s - \sqrt{-\partial_{xx}})(\partial_s + \sqrt{-\partial_{xx}}).\end{aligned} \tag{46}$$

  The space of analytic signals $H_2(\partial\mathbb{C}^+)$ is very special since its elements are treated similarly by the operators $-\sqrt{-\Delta}$ and $i\partial_x$:

$$-\sqrt{-\partial_{xx}}f = i\partial_x f \quad \text{for } f \in H_2(\partial\mathbb{C}^+),$$

  which can be easily be verified in the Fourier domain. Consequently for sufficiently smooth[19] f ($f \in \mathbb{H}_\infty$):

$$\begin{aligned}u(x,s) &= (H_s * f)(x) = (e^{-s\sqrt{-\partial_{xx}}}f)(x)\\ &= (e^{s\,i\partial_x}f)(x) = \tilde{u}(x+is).\end{aligned}$$

- The extension of the Gaussian semigroup restricted to the positive imaginary axis corresponds to the Schrödinger semigroup of the free particle. Let P be the restriction of the extension of the Poisson semigroup Q to the positive imaginary axis, i.e. $P_t = Q_{it}, t > 0$, then the restriction of P to the analytic signal subspace $H^2(\partial\mathbb{C}^+)$ equals the positive wavefront semigroup restricted to $H^2(\partial\mathbb{C}^+)$ and the restriction of P to $(H^2(\partial\mathbb{C}^+))^{\perp}$ equals the negative wavefront semigroup restricted to $(H^2(\partial\mathbb{C}^+))^{\perp}$. Since analogue to (46) we have:

$$\begin{aligned}u_{tt} - u_{xx} &= (\partial_t - \partial_x)(\partial_t + \partial_x)u\\ &= (\partial_t - i\sqrt{-\partial_{xx}})(\partial_t + i\sqrt{-\partial_{xx}})u.\end{aligned}$$

Complex analytic extension can only be done in the signal case ($d = 1$). For images $d \geq 2$ an analogue recipe can be followed, using the more general notion of Clifford analytic functions. To this end some knowledge of Clifford algebra is necessary, cf. [7, 16]. Let $\{\mathbf{e}_i\}_{i=1}^n = \{\mathbf{e}_i\}_{i=1}^d \cup \{\mathbf{e}_{d+1}\}$, $n = d+1$, be an orthonormal base in $\mathbb{R}^n$ and let $\mathbb{R}_n$ and $\mathbb{R}_n^+$ be the Clifford algebra and its even subalgebra of $\mathbb{R}^n$. Let Ω be an open set in $\mathbb{R}^n$.

Definition 9. A function $\tilde{u} \in C^\infty(\Omega, \mathbb{R}_n^+)$ is Clifford analytic on Ω if

$$\nabla_n \tilde{u} = \sum_{j=1}^{n} \mathbf{e}_j \frac{\partial \tilde{u}}{\partial x^j} = 0.$$

There again exists a (generalized) Cauchy integral theorem for these functions, cf. [16], p. 103. Analogue to the $d = 1$ case we define the closed subspace of $\mathbb{L}_2(\mathbb{R}^d)$:

$$H^2(\partial\mathbb{R}_n^+) = \{f \in \mathbb{L}_2(\mathbb{R}^d) \mid (I - \mathbf{R}\mathbf{e}_{d+1})f = 0\}.$$

Notice that $(R_j f, f) = (\mathcal{F}(R_j f), \mathcal{F} f) = 0$ for $j = 1 \ldots d$ and $(\mathbf{R})^2 = \sum R_j^2 = -I$, therefore we can split complex valued signals into a Clifford analytic and orthogonal to Clifford analytic part:

$$\begin{aligned} \mathbb{L}_2(\mathbb{R}^d) &= H^2(\partial\mathbb{R}_n^+) \oplus (H^2(\partial\mathbb{R}_n^+))^\perp \\ f &= \frac{f + \mathbf{R}\mathbf{e}_{d+1} f}{2} + \frac{f - \mathbf{R}\mathbf{e}_{d+1} f}{2} \\ &= f_{AN} + f_{NAN}, \end{aligned}$$

Notice that these two subspaces of $\mathbb{L}_2(\mathbb{R}^d)$ are precisely the irreducible subspaces of the semi-direct product of the dilation and translation group on $\mathbb{R}^d$ and that $\frac{I+\mathbf{R}\mathbf{e}_{d+1}}{2}$ and $\frac{I-\mathbf{R}\mathbf{e}_{d+1}}{2}$ are the orthogonal projections on them.

We define the Cauchy operator $C : \mathbb{L}_2(\mathbb{R}^d) \to H^2(\mathbb{R}_n^+)$ by

$$(Cf)(\mathbf{x}, s) = \frac{1}{\sigma_{d+1}} \int_{\mathbb{R}^d} \frac{\mathbf{z} - \mathbf{u}}{\|\mathbf{z} - \mathbf{u}\|^{d+1}} \mathbf{e}_{d+1} f(\mathbf{u})\, d\mathbf{u},$$

$\mathbf{z} = \sum_{j=1}^{d} x^j \mathbf{e}_j + s\mathbf{e}_{d+1}$, which can again be expressed in the Poisson kernel and its harmonic conjugate:

$$\begin{aligned} \mathbf{Q}_s(\mathbf{x}) &= \mathbf{R} H_s(\mathbf{x}) \\ &= \sum_{j=1}^{d} \mathbf{e}_j R_j H_s(\mathbf{x}) \\ &= \sum_{j=1}^{d} \mathbf{e}_j Q_s^{(j)}(\mathbf{x}) \\ &= \sum_{j=1}^{d} \frac{2}{\sigma_{d+1}} \frac{x^j \mathbf{e}_j}{(s^2 + \|\mathbf{x}\|^2)^{\frac{d+1}{2}}}, \end{aligned}$$

by

$$\begin{aligned} (Cf)(\mathbf{x}, s) &= \frac{1}{2}(H_s * f)(\mathbf{x}) \\ &\quad + \frac{1}{2} \sum_{j=1}^{d} \mathbf{e}_j \mathbf{e}_{d+1} \big(Q_s^{(j)} * f\big)(\mathbf{x}) \\ &= \left(H_s * \left(\frac{1}{2}(I + \mathbf{R}\mathbf{e}_{d+1})\right) f\right)(\mathbf{x}) \\ &= (H_s * f_{AN})(\mathbf{x}). \end{aligned}$$

Remarks

- The nil-space of C equals $(H^2(\partial\mathbb{R}_n^+))^\perp$, so $Cf = C(f_{AN} + f_{NAN}) = C(f_{AN})$.
- Let $d = 3$ and $\tilde{u}$ be Clifford analytic, then $\nabla_d \tilde{u} = 0$ and therefore

  $$\nabla_d \tilde{u}\mathbf{e}_{d+1} = \nabla_d \cdot (\tilde{u}\mathbf{e}_{d+1}) + \nabla_d \wedge (\tilde{u}\mathbf{e}_{d+1}) = 0 + \mathbf{0}$$

  so if we put $\mathbf{u} = \tilde{u}\mathbf{e}_{d+1}$ we have rot $\mathbf{u} = \mathbf{0}$ and div $\mathbf{u} = 0$ from which it follows that $\mathbf{u}$ has a harmonic potential $\mathbf{u} = \nabla p$, with $\Delta p = 0$.
- The monogenic scale space $\mathbf{u}_M$ which is introduced by Felsberg and Sommer, cf. [7] (for $d = 2$) is given by

  $$\begin{aligned} \mathbf{u}_M(\mathbf{x}, s) &= \tilde{u}(\mathbf{x}, s)\mathbf{e}_{d+1} = 2(Cf)(\mathbf{x}, s)\mathbf{e}_{d+1} \\ &= \left(H_s + \sum_{j=1}^{d} \mathbf{e}_j \mathbf{e}_{d+1} R_j H_s\right) * \mathbf{f}) \\ &= \mathbf{e}_{d+1}(H_s * f)(\mathbf{x}) + \sum_{j=1}^{d} \mathbf{e}_j \big(Q_s^{(j)} * f\big)(\mathbf{x}), \end{aligned}$$

  where $\mathbf{f} = f\mathbf{e}_{d+1}$. By Eqs. (36) and (38) it follows that the other components in the monogenic scale space besides the Poisson scale space describe the Poisson image flow analogue to the fact that $-\nabla u$ describes the Gaussian image flow.

Some interesting local features can easily be obtained from the Monogenic/Clifford analytic scale spaces, such as the local phase $\mathbf{r}$, vector field, local attenuation A (amplitude e^A), local orientation ($\frac{\mathbf{r}}{\|\mathbf{r}\|}$). These concepts are again generalizations of the local phase analysis in signal analysis and are related by the logarithm $(A, \mathbf{r}) = \log(u, \mathbf{R}u) = \log\sqrt{|u|^2 + \|\mathbf{R}u\|^2} + \frac{\mathbf{R}u}{\|\mathbf{R}u\|} \arctan \frac{\|\mathbf{R}u\|}{u} \Leftrightarrow (u, \mathbf{R}u) = e^{A\mathbf{r}} = e^A(\frac{\mathbf{r}}{\|\mathbf{r}\|} \sin\|\mathbf{r}\|, \cos\|\mathbf{r}\|)$ of the Monogenic scale space, cf. [7, 8, 10].

6. Fractional Powers of Closed Operators

This section gives a short introduction into the theory of fractional powers of positive, closed and self adjoint operators. First, we deal with some general theory and then we apply it to the special case $\mathcal{A} = \Delta$. For more details the reader is referred to Yosida [39], Rudin [33] and Balakrishnan [2].

For every positive, closed self-adjoint operator (not necessarily bounded) $-\mathcal{A} > 0$ on a Hilbert space H there exists a unique self-adjoint $\mathcal{B}$, such that $\mathcal{B}^2 = -\mathcal{A}$. This can be shown by using a resolution of the identity E on the Borel subsets of the real line such that $(\mathcal{A}x, y) = \int_0^\infty s\, dE_{x,y}(s)$, which exists since $-\mathcal{A}$ is self adjoint. Operator $\mathcal{B}$ is then uniquely defined by

$$(\mathcal{B}x, y) = \int_0^\infty \sqrt{s}\, \mathrm{dE_{x,y}(s)}. \tag{47}$$

See, Rudin [33] Theorem 13.31 and 13.30.

In this report we write $\sqrt{-A}$ or $-\mathcal{A}^{\frac{1}{2}}$ in stead of $\mathcal{B}$. Notice that *if* $-\mathcal{A}$ is also compact, $\sigma(-\mathcal{A})$ is discrete and $-\mathcal{A}$ has a complete set of eigenfunctions. Then operator $\sqrt{-\mathcal{A}}$ is uniquely determined by taking square roots of the eigen values. This is exactly what happens if one deals with α scale spaces on a finite domain, with Neumann boundary conditions! Then the α scale spaces are very directly related to the Gaussian scale space simply by taking α powers of the minus eigenvalues of the diffusion generator, i.e. the solution of the α scale space on a finite domain with Neumann boundary conditions takes the simple form $u_\alpha = \sum_n (f_n, f) f_n e^{-(-\lambda_n)^\alpha s}$, cf. [5]. However in this article we focus on the infinite $\mathbb{R}^d$ case in which it takes quite an effort to derive the direct relation between the α scale spaces as we will see.

Given a strongly continuous semigroup Q with infinitesimal generator $\mathcal{A}$ one can construct a *holomorphic* semigroup $\hat{Q}_\alpha (0 < \alpha < 1)$, such that the corresponding infinitesimal generator $\hat{\mathcal{A}}_\alpha$ equals $-(-\mathcal{A})^\alpha$, which is defined by (61), see Theorem 10. In Theorem 10 we will show that these operators indeed satisfy $(-\mathcal{A})^\alpha(-\mathcal{A})^\beta = (-\mathcal{A})^{\alpha+\beta}$ for $\alpha + \beta < 1$ and $\alpha = \beta = \frac{1}{2}$. From this we conclude that the square root of $-\mathcal{A}$ is indeed *explicitly* given by (61) setting $\alpha = \frac{1}{2}$. We will apply this fundamental theory to the case $\mathcal{A} = \Delta$ and thereby construct the α scale spaces from their "mother" scale space; the Gaussian scale space, which at the same time gives the strong connection between all α-scale spaces. First we will do some preparation before we obtain this fundamental result.

For $s, \sigma > 0$, $0 < \alpha < 1$ we define $q_{s,\alpha} : \mathbb{R} \to \mathbb{R}$ by[20]

$$q_{s,\alpha}(\lambda) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} e^{z\lambda - s z^\alpha}\, \mathrm{dz} & \text{if } \lambda \geq 0 \\ 0 & \text{else} \end{cases} \tag{48}$$

where the branch of $z^\alpha = e^{\alpha \ln z} = |z|^\alpha e^{i\alpha \arg(z)}$ is the one valued function in the z-plane cut along the negative real axis. By Cauchy's integration theorem it immediately follows that (48) is independent of the choice of $\sigma = \Re(z) > 0$. By deforming the path of integration in (48) to the union of two paths $re^{i\theta}$, $re^{-i\theta}$, for *any* fixed $\theta \in \frac{\pi}{2}$ and r running from 0 to infinity we obtain by straightforward computation

$$q_{s,\alpha}(\lambda) = \frac{1}{\pi} \int_0^\infty \sin(\lambda r \sin\theta - s\, r^\alpha \sin(\alpha\theta) + \theta) * e^{\lambda r \cos\theta - s\, r^\alpha \cos(\alpha\theta)}\, \mathrm{dr} \tag{49}$$

Yosida [39] uses this formula, in particular the case $\theta = \theta_\alpha = \frac{\pi}{1+\alpha}$, frequently in the proofs in paragraph $IX - 11$ of his book. In order to avoid these nasty calculations we will use Laplace transform properties of $q_{t,\alpha}$ instead. The Laplace Transform $\mathcal{L}(q_{s,\alpha})$ of $q_{s,\alpha}$ is given by

$$\mathcal{L}(q_{s,\alpha})(\mu) = e^{-s\mu^\alpha} \quad \mu > 0, \tag{50}$$

since by Cauchy's theorem of residue after choosing $\sigma < s$ we have

$$\int_0^\infty e^{-\lambda\mu} q_{s,\alpha}(\lambda)\, \mathrm{d}\lambda = \frac{-1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{z - \mu} e^{-z^\alpha s}\, \mathrm{dz} = e^{-s\mu^\alpha}.$$

As a result we have

$$q_{s+t,\alpha} = q_{s,\alpha} \star q_{t,\alpha} \quad s, t > 0, \tag{51}$$

where $\star$ represents the convolution product with respect to Laplace transform. Moreover, by (50) it follows that:

$$\mathcal{L}(q_{1,\alpha})(s^{1/\alpha}\mu) = \mathcal{L}(q_{s,\alpha})(\mu) \quad s, \mu > 0$$

and therefore[21]

$$s^{-\frac{1}{\alpha}}\, q_{1,\alpha}(s^{-1/\alpha}\lambda) = q_{s,\alpha}(\lambda)\ s > 0 \tag{52}$$

By differentiating (50) with respect to s we have

$$\mathcal{L}(q'_{s,\alpha})(\mu) = -\mu^{\alpha} e^{-s\mu^{\alpha}} \tag{53}$$

Consequently,

- $\mathcal{L}(q'_{1,\alpha})(\mu\, s^{1/\alpha}) = s\ \mathcal{L}(q'_{s,\alpha})(\mu)$, $s > 0$, which gives us the following equation

$$s^{-1-1/\alpha} q'_{1,\alpha}(s^{-1/\alpha}\lambda) = q'_{s,\alpha}(\lambda) \quad \lambda, s > 0. \tag{54}$$

- Let $s \downarrow 0$ in (53) we easily obtain

$$\lim_{s\downarrow 0} q'_{s,\alpha}(\lambda) = -\frac{\lambda^{-\alpha-1}}{\Gamma(-\alpha)}. \tag{55}$$

  Since,

$$\mathcal{L}(\lambda^{-1-\alpha})(\mu) = \mu^{\alpha}\left(\mu^{-\alpha}\int_0^{\infty} \lambda^{-\alpha-1} e^{-\mu\lambda}\, d\lambda\right) = \mu^{\alpha}\Gamma(-\alpha). \tag{56}$$

  Note that $0 < \alpha < 1$, the Gamma function can be extended analytically to $\mathbb{C} \setminus \mathbb{Z}^-$ by $\Gamma(z) = \frac{\Gamma(z+n)}{(z+n-1)(z+n-2)\cdots z}$, $-n \leq \Re(z) \leq 0$.

It can be shown (see Yosida [39], p. 261) that

$$q_{s,\alpha}(\lambda) = \lim_{n\to\infty} \frac{(-1)^n}{n}\left(\frac{n}{\lambda}\right)^{n+1} \mathcal{L}^{(n)}(q_{s,\alpha})\left(\frac{n}{\lambda}\right), \lambda > 0.$$

Thereby,

$$q_{s,\alpha}(\lambda) \geq 0 \quad \text{for all } \lambda > 0. \tag{57}$$

Obviously we have $\int_0^{\infty} q_{s,\alpha}(\lambda)\, d\lambda < \infty$, for instance by (49). Further we have $|q_{s,\alpha}(\lambda)e^{-\mu\lambda}| \leq |q_{s,\alpha}(\lambda)| = q_{s,\alpha}(\lambda)$, so by Lebesgue's dominated convergence principle and (50) we have

$$\int_0^{\infty} q_{s,\alpha}(\lambda)\, d\lambda = \int_0^{\infty} \lim_{\mu\downarrow 0} q_{s,\alpha}(\lambda) e^{-\mu s}\, d\lambda = \lim_{\mu\downarrow 0} \mathcal{L}(q_{s,\alpha})(\mu) = 1 \tag{58}$$

As a result, since $q_{s,\alpha}$ is differentiable with respect to s, we have

$$\int_0^{\infty} \frac{\partial q_{s,\alpha}(\lambda)}{\partial s}\, d\lambda = 0, \quad s > 0. \tag{59}$$

Let Q be a semigroup on a Banach space X, with infinitesimal generator $\mathcal{A}$. Define for $1 > \alpha > 0$ the operator $\hat{Q}_{s,\alpha} : X \to X$ by

$$\hat{Q}_{s,\alpha} x = \int_0^{\infty} q_{s,\alpha}(\eta) Q_{\eta} x\, d\eta \quad (s > 0). \tag{60}$$

Theorem 10. *Let Q be a strongly continuous semigroup, with infinitesimal generator $\mathcal{A}$ on a Banach space X such that $\mathcal{A}$ has a resolvent $R(\lambda; \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ for all $\lambda > 0$. Let $\hat{Q}_{s,\alpha} : X \to X$ be given by* (60).

Then, the mapping $s \to \hat{Q}_{s,\alpha}$ is a holomorphic semigroup on X, which infinitesimal generator $-\hat{\mathcal{A}}_{\alpha}$ satisfies

$$\hat{\mathcal{A}}_{\alpha} = (-\mathcal{A})^{\alpha},$$

where $(-\mathcal{A})^{\alpha}$ is the operator on X given by

$$(-\mathcal{A})^{\alpha} x = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty} \lambda^{\alpha-1}(\lambda I - \mathcal{A})^{-1}(-\mathcal{A}x)\, d\lambda, \tag{61}$$

for $x \in \mathcal{D}(\mathcal{A})$. Moreover, if $\sup_{\Re(\lambda)>0} |\Re(\lambda)| \cdot \|R(\lambda; \mathcal{A})\| < \infty$, then

$$\begin{aligned} &(-\mathcal{A})^{\alpha}(-\mathcal{A})^{\beta} = (-\mathcal{A})^{\alpha+\beta} \qquad 0 < \alpha + \beta < 1 \\ &\lim_{\alpha\uparrow 1} (-\mathcal{A})^{\alpha} x = (-\mathcal{A})x \quad \text{if } x \in \mathcal{D}(\mathcal{A}) \\ &\lim_{\alpha\downarrow 0} (-\mathcal{A})^{\alpha} x = x \quad \text{if } \lim_{\lambda\downarrow 0} \lambda\, R(\lambda; \mathcal{A})x = 0. \end{aligned} \tag{62}$$

Proof: First we will show that $s \to \hat{Q}_{s,\alpha}$ is a strongly continuous semigroup for every $\alpha > 0$. Let $\alpha \in (0, 1)$, $x \in X$ and $s, t > 0$. Then, by (60) and (51) we have

$$\begin{aligned} \hat{Q}_{s,\alpha}(\hat{Q}_{t,\alpha} x) &= \int_0^{\infty}\int_0^{\infty} q_{s,\alpha}(\eta) q_{t,\alpha}(\xi) Q_{\xi+\eta} x\, d\xi\, d\eta \\ &= \int_0^{\infty} (q_{s,\alpha} \star q_{t,\alpha})(\sigma) Q_{\sigma} x\, d\sigma \\ &= \int_0^{\infty} (q_{s+t,\alpha})(\sigma) Q_{\sigma} x\, d\sigma \\ &= \hat{Q}_{s+t,\alpha} x. \end{aligned}$$

So, $\hat{Q}_{s+t,\alpha} = \hat{Q}_{s,\alpha}\hat{Q}_{t,\alpha}$. Further, by (52) and by substitution $\xi = s^{-1/\alpha}\eta$ we obtain

$$\begin{aligned} \hat{Q}_{s,\alpha} x &= \int_0^{\infty} s^{-1/\alpha} q_{1,\alpha}(s^{-1/\alpha}\eta)\, Q_{\eta} x\, d\eta \\ &= \int_0^{\infty} q_{1,\alpha}(\xi)\, Q_{\xi s^{1/\alpha}} x\, d\xi. \end{aligned} \tag{63}$$

Now from Lebesgue's dominated convergence principle it follows that

$$\lim_{s\downarrow 0} \|\hat{Q}_{s,\alpha}x - x\|_X = \int_0^\infty q_{1,\alpha}(\xi)\left\{\lim_{s\downarrow 0}\|Q_{\xi s^{1/\alpha}}x - x\|\right\} \mathrm{d}\xi$$

which tends to 0 as $s \downarrow 0$. Second, we will show that this semigroup is holomorphic. By (54) and by substitution $\xi = s^{-1/\alpha}\lambda$ it follows that

$$\begin{aligned}\hat{Q}_s' x &= \int_0^\infty q_{s,\alpha}'(\lambda) Q_\lambda x \,\mathrm{d}\lambda \\ &= \frac{1}{s}\int_0^\infty q_{1,\alpha}'(\xi)\, Q_{s^{1/\alpha}\xi} x \,\mathrm{d}\xi.\end{aligned}$$

Due to the uniform boundedness principle (see Theorem 2.5.2 [18]) there exists a $M > 0$ such that $\|Q_s\| \le M$ for all $s \le 1$. Therefore,

$$\|sQ_s'\| \le M \int_0^\infty |q_{1,\alpha}'(\xi)| \,\mathrm{d}\xi < \infty.$$

So by (23) we have that $\hat{Q}$ is indeed a holomorphic semigroup.

Third, we show that the infinitesimal generator of this semigroup indeed equals $-\mathcal{A}^\alpha$. Using (55) and (59) we obtain

$$\begin{aligned}\hat{\mathcal{A}}_\alpha x = \hat{Q}_0' x &= \lim_{s\downarrow 0}\int_0^\infty q_{s,\alpha}'(\lambda)\; Q_\lambda x \,\mathrm{d}\lambda \\ &= \lim_{s\downarrow 0}\int_0^\infty q_{s,\alpha}'(\lambda)\,(Q_\lambda - I)x \,\mathrm{d}\lambda \\ &= -\frac{1}{\Gamma(-\alpha)}\int_0^\infty \lambda^{-\alpha-1}(Q_\lambda - I)x \,\mathrm{d}\lambda,\end{aligned} \tag{64}$$

for $x \in \mathcal{D}(\mathcal{A})$. We will rewrite this expression by using (56) with $1+\alpha$ in stead of $-\alpha$, Theorem 4 and the formula $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$:

$$\begin{aligned}\hat{A}_\alpha x &= \frac{1}{\Gamma(-\alpha)\Gamma(1+\alpha)} \\ &\quad\times \int_0^\infty \left\{\int_0^\infty e^{-\lambda t} t^\alpha \,\mathrm{d}t\right\}(I - Q_\lambda)\,x\; d\lambda \\ &= \frac{\sin\alpha\pi}{\pi}\int_0^\infty t^\alpha\left((tI - \mathcal{A})^{-1} - t^{-1}I\right)x \,\mathrm{d}t \\ &= \frac{\sin\alpha\pi}{\pi}\int_0^\infty t^{\alpha-1}(tI - \mathcal{A})^{-1}\mathcal{A}x \,\mathrm{d}t\end{aligned}$$

for $x \in \mathcal{D}(\mathcal{A})$. Since a formal proof of (62) yields much computation we will skip the proof. For a proof see Yosida [39] p. 267.

Nevertheless, we will show this for the special case that $\mathcal{A}$ is self-adjoint. Notice that by assumption $\mathcal{A}$ must be negative definite. From (61) it follows that $\hat{\mathcal{A}}_\alpha$ is also self adjoint, commuting with $\mathcal{A}$. So with the eye on the spectral resolution (47) and the fact that $\mathcal{A}$ and $\hat{\mathcal{A}}_\alpha$ are commuting self-adjoint operators on a Banach space, it follows that we only need to show (62) for the case that x equals an eigenfunction with eigenvalue $-\mu$, $\mu > 0$ (with respect to $\mathcal{A}$). We shall use the formula

$$\begin{aligned}\int_0^\infty \frac{v^{p-1}}{1+v}\,\mathrm{dv} &= \mathrm{Beta}(p, 1-p) = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)} \\ &= \frac{\pi}{\sin\pi p},\; 0 < \Re(p) < 1.\end{aligned} \tag{65}$$

This integral is convergent and analytic (as a function of p) and can be calculated by a "Pac-Man" contour around the real axis in the complex plane or by substitution of $t = \frac{x}{1+x}$ in Euler's beta function.

By straightforward computation and (65) we have

$$\begin{aligned}-(-\mathcal{A})^\alpha x &= -\frac{\sin\alpha\pi}{\pi}\int_0^\infty \frac{t^{\alpha-1}}{t+\mu}\mu x \,\mathrm{d}t \\ &\quad - \mu^\alpha\frac{\sin\alpha\pi}{\pi}\int_0^\infty \frac{v^{p-1}}{1+v}x \,\mathrm{d}v = -\mu^\alpha \mathrm{x},\end{aligned}$$

$x \in E_\mu(\mathcal{A}), (v = \frac{t}{\mu})$. So, $(-\mathcal{A})^\alpha(-\mathcal{A})^\beta = \mathcal{A}^{\alpha+\beta}$, for $\alpha + \beta < 1$. Moreover, we have $\lim_{\alpha\uparrow 1}(-\mathcal{A})^\alpha x = \lim_{\alpha\uparrow 1} -\mu^\alpha x = -\mu x = -\mathcal{A}x$ and $\lim_{\alpha\downarrow 0} -(-\mathcal{A})^\alpha x = x$. □

Remark. In the special case where $\mathcal{A}$ self-adjoint and negative definite we also have $(-(-\mathcal{A})^{1/2})^2 = -\mathcal{A}$. Take $\alpha = \beta = 1/2 - 1/n$, $n \in \mathbb{N}$ in (62) and let $n \to \infty$.

Although in the general case the eigenvalues need not be real-valued (62) remains valid, if $\alpha + \beta < 1$. If x is an eigenfunction of $-\mathcal{A}$ with eigenvalue μ. One might think that this condition is due to the convergence of (65), but probably this is not true! To this end we notice that the infinity of the integral is due to the representation of $-(-\mathcal{A})^\gamma x$ by (61), which is valid for $0 < \gamma < 1$.

The true essence of the restriction $\alpha + \beta < 1$ is that the formula

$$z^{\alpha+\beta} = z^\alpha z^\beta \quad \text{with } z \in \mathbb{C} \text{ such that } \Re(z) > 0$$

is only valid for $(0 <)\alpha + \beta < 1$, since the argument of $z^{\alpha+\beta}$ may not exceed the negative axis cut in the complex plane. Note that by the assumptions of Theorem 10 the real part of eigenvalues of the operator $-\mathcal{A}$ in this theorem must be positive.

This problem doesn't arise in the self adjoint case where all eigenvalues are real valued.

6.1. *Fractional Powers of the Minus Laplace Operator:* $(-\Delta)^\alpha$, $0 < \alpha \leq 1$

By Green's first respectively second identity it follows that Δ is a negative definite respectively self adjoint on the vector space of twice continuously differentiable functions. This vector space is dense in $\mathbb{H}_2(\mathbb{R}^d)$ and since Δ is a closed operator Δ is also self adjoint and positive on the Banach space $\mathbb{H}_2(\mathbb{R}^d$. Therefore we can apply Theorem 10 to the Gaussian semigroup $Q : \mathbb{R}^+ \to \mathcal{B}(\mathbb{H}_2(\mathbb{R}^d))$ and obtain a new holomorphic (!) semigroup $\hat{Q} : \mathbb{R}^+ \to \mathcal{B}(\mathbb{H}_{2\alpha}(\mathbb{R}^d))$ with infinitesimal generator $-(-\Delta)^\alpha$ which satisfies

$$\hat{Q}_s f = \int_0^\infty q_{t,\alpha}(\xi) Q_\xi f \, \mathrm{d}\xi \quad \mathrm{f} \in \mathbb{L}_2(\mathbb{R}^\mathrm{d}),$$
$$-(-\Delta)^\alpha f = \frac{\alpha\pi}{\pi} \int_0^\infty t^{\alpha-1} R(t;\Delta)\Delta f \, \mathrm{dt} \quad \mathrm{f} \in \mathbb{H}_2(\mathbb{R}^\mathrm{d}).$$

6.2. *Derivation of the Poisson Semigroup from the Gaussian Semigroup*

The special case $\alpha = \frac{1}{2}$ leads to the Poisson semigroup. Since by equality (49), for the special case $\theta = \pi$, we have

$$q_{s,1/2}(\xi) = \frac{1}{\pi} \int_0^\infty e^{-\xi r} \sin(s\sqrt{r}) \, \mathrm{dr} = \frac{8\mathrm{s}}{\xi\sqrt{\pi\xi}} \mathrm{e}^{-\mathrm{s}^2/4\xi}.$$

So the Poisson semigroup is given by:

$$\begin{aligned}
(\hat{Q}_s f)(\mathbf{u}) &= \int_0^\infty q_{s,1/2}(\eta) Q_\eta f \, \mathrm{d}\eta \\
&= \int_0^\infty \frac{8s}{\eta\sqrt{\pi\eta}} e^{-\frac{s^2}{4\eta}} \int_{\mathbb{R}^d} \frac{e^{-\frac{\|\mathbf{u}-\mathbf{v}\|^2}{4\eta}}}{(4\pi\eta)^{d/2}} f(\mathbf{v}) \, \mathrm{d}\mathbf{v} \, \mathrm{d}\eta \\
&= \frac{2^{(3-d)}}{\pi^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} \left(\int_0^\infty \frac{e^{-\frac{\|\mathbf{u}-\mathbf{v}\|^2}{4\eta}}}{\eta^{(d+1)/2}} \, \mathrm{d}\eta \right) f(\mathbf{v}) \mathrm{d}\mathbf{v} \\
&= \frac{2}{\sigma_{d+1}} \int_{\mathbb{R}^d} \frac{s}{(s^2 + \|\mathbf{u}-\mathbf{v}\|^2)^{\frac{d+1}{2}}} f(\mathbf{v}) \, d\mathbf{v} \\
&= (H_s * f)(\mathbf{u}) \quad \mathbf{u} \in \mathbb{R}^d.
\end{aligned}$$

which is indeed a convolution with the Poisson kernel.

6.3. *Verification of the Axioms with Respect to the* α *Scale Spaces*

Let us consider the scale evolution system

$$\begin{cases} \dfrac{\partial u}{\partial s} = -(-\Delta)^\alpha u & (0 < \alpha < 1), \\ \lim_{s \downarrow 0} u(\mathbf{x}, s) = f(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^d). \end{cases} \tag{66}$$

Since we already verified the axioms in the Poisson case ($\alpha = 1/2$), we will only highlight the axioms for which a generalization to the case $\alpha \in (0, 1)$ is not trivial. Notice that we have already shown in Section 3 that all α-scale spaces obey the weak causality principle. They do not satisfy the Koenderinks principle nor the special cylinder maximum principle nor the strong causality principle. Felsberg [10] used a signal consisting of three delta spikes to show that Koenderinks principle is sometimes not satisfied in Poisson scale space. We use the same example to illustrate the evolution of isophotes as α increases (see Fig. 3). Next we derive a formal expression for the α convolution kernel in the spatial domain:

$$\begin{aligned}
u(\mathbf{x}, s) = [\hat{Q}_s f](\mathbf{x}) &= \int_0^\infty q_{s,\alpha}(t)(G_t * f)(\mathbf{x}) \, \mathrm{dt} \\
&= \left(\left[\int_0^\infty q_{s,\alpha}(t) G_t \, \mathrm{dt} \right] * f \right)(\mathbf{x}), \alpha \in (0, 1).
\end{aligned} \tag{67}$$

So the convolution kernel equals $\int_0^\infty \frac{q_{s,\alpha}(t)}{\sqrt{4\pi t}} e^{\frac{-\|\mathbf{x}\|^2}{4t}}$ dt. Since $q_{s,\alpha}$ is positive cf. (57) and $\|q_{s,\alpha}\|_{\mathbb{L}_1(\mathbb{R}^d)} = 1$ we have by Lebesgue's dominated convergence principle (applied on $\{\int_0^N q_{s,\alpha}(t) G_t \, \mathrm{dt}\}_{\mathrm{N}\in\mathbb{N}}$):

$$\begin{aligned}
\left\| \int_0^\infty q_{t,s} K_t \, \mathrm{dt} \right\|_{\mathbb{L}_1(\mathbb{R}^d)} &= \int_0^\infty q_{t,s} \|K_t\|_{\mathbb{L}_1(\mathbb{R}^d)} \, \mathrm{dt} \\
&= \int_0^\infty q_{t,s} \, \mathrm{dt} = 1
\end{aligned}$$

So indeed average grey value is preserved, when using this filter. It also follows from (67) and the positivity of $q_{s,\alpha}$ that Axiom 4 is satisfied. See also (11). It follows directly from (52) and (67) that Axiom 3 is satisfied. It follows from (67) and the rotation invariance of the Gaussian kernel that the rotation invariance property is also satisfied. In a analogue matter we have that the convolution kernel has a finite $\mathbb{L}_2$-norm, therefore by Cauchy Schwarz it follows that Q_s is a bounded operator from $\mathbb{L}_2(\mathbb{R}^d)$ into $\mathbb{L}_\infty(\mathbb{R}^d)$.

With regard to Axiom 11 we remark that until now a generalization of Theorem 8 hasn't been found. Notice that it is quite hard to compare the α scale space since the scale parameters have different physical dimension $[\text{LENGTH}]^{2\alpha}$. Moreover, the α scale spaces for $\alpha < 1$ do not have finite variance, which directly follows from (11). Variance is *not* a good general measure for kernel width. This is only true for the Gaussian case. The comparison of α scale spaces on a bounded domain with Neumann boundary conditions is (in comparison to the unbounded domain considered in this article) much more obvious using the notion of relative scale, cf. [5].

7. Gaussian Filtering and Poisson Filtering

In this section we mainly focus on special properties (besides the mentioned axioms) of Gaussian filtering and investigate whether similar results can be obtained using Poisson filtering. It will turn out that the derivatives of the Poisson kernels have at least as nice properties as the derivatives of the Gaussian kernels. At the end of this section we will present some practical results using different members of the unique class of filters, satisfying all Axioms. The following figure shows the similarity between the Gaussian and Poisson kernel:

7.1. Using One Dimensional Kernels in Multi-Dimensional Implementation

The Gaussian kernel G_s is separable and even satisfies

$$G(x_1, \ldots, x_d) = \prod_{i=1}^{d} G_s^{(1)}(x_i), \tag{68}$$

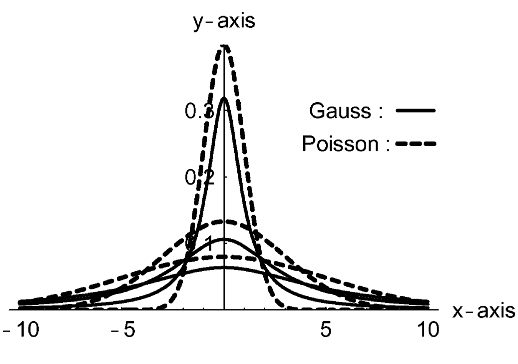


Figure 4. The graphs of Gaussian kernels $\sigma = 1, 3, 5$ and Poisson for $s = 1, 3, 5$.

with $G^{(1)}$ the 1D Gauss kernel. Of course this is a very nice property with respect to multi-dimensional implementation in the spacial domain, since computation becomes order $O(dn)$ in stead of $O(n^d)$. It is easy to see that the Poisson kernel H_s does not have this property. Namely, suppose $H_s(x_1, \ldots, x_d) = \prod_{j=1}^{d} f_s^j(x_j)$, then we would have

$$\frac{\frac{\partial}{\partial x_i} H_s(x_1, \ldots, x_d)}{H_s(x_1, \ldots, x_d)} = \frac{\left[f_s^i\right]'(x_i)}{f_s^i(x_i)} \quad \text{for all } i = 1 \ldots d,$$

with the righthand side depending only on x_i, but clearly the left hand side depends on $x_1, \ldots, x_d$. The only C^1 filter Φ which is both separable $\Phi(x, y) = \phi(x)\psi(y)$ and isotropic must be Gaussian, since

$$\begin{aligned} \mathcal{P}_R \Phi &= \Phi \text{ for all } R \in SO(d) \\ &\Leftrightarrow (y\partial_x - x\partial_y)\Phi(x, y) = 0 \\ &\Leftrightarrow \frac{y\psi(y)}{\psi'(y)} = \frac{x\phi(x)}{\phi'(x)} = C. \end{aligned} \tag{69}$$

The separability of Gaussian kernels coincides with the fact that the 1D infinitesimal generators (∂_i^2) commute and satisfy

$$\Delta = \sum_{i=1}^{d} \partial_i^2.$$

Although the 1D infinitesimal generators of 1D Poisson semigroups commute they do not satisfy the last property:

$$-\sqrt{-\Delta} \neq \sum_{i=1}^{d} -\sqrt{-\partial_i^2},$$

but they do satisfy

$$(-\sqrt{-\Delta})^2 = -\Delta = -\sum_{i=1}^{d} \partial_i^2 = \sum_{i=1}^{d} \left(-\sqrt{-\partial_i^2}\right)^2. \tag{70}$$

Suppose we would write in case $d = 3$:

$$-\sqrt{-\Delta} = \sum_{i=1}^{3} -\mathbf{e}_i \sqrt{-\partial_i^2},$$

with $\{\mathbf{e}_i\}$ a righthanded orthonormal base in a 3D euclidian space. Then it follows by (70) that the $\mathbf{e}_i$ must satisfy

$$\frac{1}{2}(\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i) = \delta_{ij} \quad i, j = 1 \ldots 3.$$

But this means that our 3-space is thus extended to the real 3DClifford/Pauli algebra

$$P = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = i\},$$

equipped with product $ab = (a, b) + a \wedge b$. Therefore the commutator with respect to this Clifford product equals 2 $a \wedge b$. Therefore, in general[22]

$$\prod_{i=1}^{d} e^{\,-\mathbf{e}_i\sqrt{-\partial_i^2}} \neq e^{-\sqrt{-\Delta}}.$$

The primitive of the Poisson kernel in contrast to the Gaussian kernel can be described explicitly. In the one dimensional case it is given by $P_s(x) = \arctan\left(\frac{x}{s}\right)$, $x > 0$. In d-dimensions it is given by

$$P_s(\mathbf{x}) = 2\frac{\sigma_d}{d\sigma_{d+1}}\left(\frac{r}{s}\right)^d F_{2,1}\left[\frac{d}{2}, \frac{d+1}{2}, \frac{2+d}{2}, -\frac{r^2}{s^2}\right].$$

Where the hypergeometric function $F_{2,1}$ is given by

$$F_{2,1}[a; b; c; z] = \sum_{k=0}^{\infty}\left(\frac{(a)_k(b)_k}{(c)_k}\right)\frac{z^k}{k!} \qquad (71)$$

and $(a)_k = \frac{\Gamma[a+k]}{\Gamma[a]}$.

For $d = 2m$ (71) can be simplified into

$$\frac{\sigma_{2m}}{\sigma_{2m+1}}(-1)^m B_{-(\frac{r}{s})^2}\left(m, \frac{1}{2} - m\right),$$

where $B_z(a; b)$ denotes the incomplete beta function.

For $d = 1$ and $d = 2$ we obtain $P_s(x) = \frac{1}{\pi}\arctan(\frac{|x|}{s})$ and $P_s(\mathbf{x}) = 1 - \frac{1}{\sqrt{1+(\frac{\|\mathbf{x}\|}{s})^2}}$.

7.2. Derivatives of the 1D Poisson and Gaussian Kernel

It is well known that the derivatives of the Gaussian kernel are given by

$$G_s^{(n)}(x) = \frac{(-1)^n}{\sigma} H_n\left(\frac{x}{\sigma}\right) G_s(x),\, x \in \mathbb{R},\, n \in \mathbb{N},$$

where H_n is the Hermite polynomial of order n. These are orthogonal polynomials, with respect to the weight function $w(x) = e^{-x^2}$ and satisfy:

$$\begin{aligned} H_n''(x) - 2xH_n'(x) + 2nH_n(x) &= 0 \\ H_n(x) &= (-1)^n e^{x^2} D^{(n)} e^{-x^2} \\ e^{2xt-t^2} &= \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n \\ H_n(x) &= n!\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k(2x)^{n-2k}}{k!(n-2k)!} \\ H_n(x) &= \frac{H_{n+1}'}{2(n+1)} \\ H_{n+2}(x) &= 2xH_{n+1}(x) - 2(n+1)H_n(x). \end{aligned}$$

A Hermite polynomial H_n is even (odd) if and only if the order n is even (odd) and it has n distinct real nil-points, which lie between the $n+1$ distinct real nil-points of H_{n+1}. The functions $\Psi_n : \mathbb{R} \to \mathbb{R}$ given by $\Psi_n(x) = e^{-\frac{x^2}{2}}H_n(x)$ form a complete orthogonal set in $\mathbb{L}_2(\mathbb{R})$ and they form a base of eigenfunctions with respect to Fourier transform, $\widehat{\Psi_n} = i^n\Psi(\omega)$.

The above results are quite useful in practice, therefore we investigate if some kind of similar results also appear in differentiating the Poisson kernel.

n times differentiation of the Poisson kernel leads to:

$$H_s^{(n)}(x) = \frac{P_{n,s}(x)}{(x^2+s^2)^{n+1}}, \qquad (72)$$

where $P_{n,s}$ is a polynomial of n-th order in x.

By differentiating both sides of (72) one easily obtains the recursion formula:

$$\begin{aligned} P_{n+1,s}(x) = {} & P_{n,s}'(x)(s^2+x^2) \\ & - 2(n+1)xP_{n,s}(x) \quad n = 0, 1, \ldots. \end{aligned} \qquad (73)$$

Next we derive a formula for the polynomials $P_{n,s}$. Write $H_s(x) = \frac{1}{2\pi i}(\frac{1}{x-is} - \frac{1}{x+is})$, then

$$H_s^{(n)}(x) = \frac{1}{2\pi i}\left[\frac{(-1)^n n!}{(x-is)^{n+1}} - \frac{(-1)^n n!}{(x+is)^{n+1}}\right],$$

so,

$$P_{n,s}(x) = \frac{(-1)^n n!}{2i\pi}[(x+is)^{n+1} - (x-is)^{n+1}]. \qquad (74)$$

Using Newton's binomium we obtain

$$P_{n,s}(x) = \frac{(-1)^n n!}{\pi} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} x^{n-2k} s^{2k+1} (-1)^k. \tag{75}$$

From this it directly follows that:

$$P'_{n,s}(x) = -(n+1)n\ P_{n-1,s}(x), \tag{76}$$

$n = 1, 2 \ldots$. It now follows by (73) and (76) that

$$P_{n+1,s}(x) + 2(n+1)x P_{n,s}(x) + n(n+1)(x^2+s^2) P_{n-1,s}(x) = 0.$$

Moreover, from (75) it follows that the polynomials $P_{n,s}$ satisfy the second order differential equation:

$$(s^2+x^2)P''_{n,s}(x) - 2nxP'_{n,s}(x) + n(n+1)P_{n,s}(x) = 0.$$

If we write $z = x + is = re^{i\theta}$, $(z \in \mathbb{C}, r > 0, 0 < \theta < \pi)$, then (74) and (72) can be written

$$\begin{cases} H_s^{(n)}(x) = \dfrac{(-1)^n}{\pi r^{n+1}} \sin(n+1)\theta = \Im\left\{\dfrac{z^{n+1}}{|z|^{2n+2}}\right\} \\ \qquad = \dfrac{(-1)^n}{2\pi i}\left[\bar{z}^{\,-(n+1)} - z^{-(n+1)}\right], \\ P_{n,s}(x) = \dfrac{(-1)^n n!}{2\pi i}\left[z^{n+1} - \bar{z}^{n+1}\right] \end{cases} \tag{77}$$

So for all $n \in \mathbb{N} \cup \{0\}$ the mapping $(\mathbf{x}, s) \mapsto P_{n,s}(\mathbf{x})$ is a homogeneous harmonic polynomial. In the sequel we will denote this mapping with Ψ_n. It satisfies[23]

$$\begin{aligned} (\Psi_n, \Psi_m)_{\mathbb{L}_2(\partial B_{\mathbf{0},R})} &= 2 * \frac{(-1)^{n+m} n! m! R^{n+m+3}}{\pi^2} \\ &\quad \times \int_0^\pi \sin(n+1)\theta \sin(m+1)\theta \, d\theta \\ &= \delta_{nm} \frac{(n!)^2 R^{2n+3}}{\pi} \end{aligned} \tag{78}$$

Note that the set consisting of the mappings $(x, s) \mapsto H_{n,s}(x)$ $n = 0, 1 \ldots$ is also orthogonal on the (upper half of the) unit circle.

Define $\chi_n : \mathbb{C} \to \mathbb{C}$ by

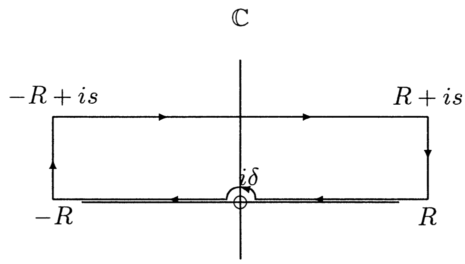
$$\chi_n(z) = \frac{(-1)^n}{2\pi i}\left[\bar{z}^{\,-(n+1)} - z^{-(n+1)}\right],$$

then by (78) we have $H_n(s) = \chi_n(x + is)$.

Since $\bar{z} = \frac{1}{z}$ on the unit disk, we have

$$\begin{aligned} &\int_{\partial B_{\mathbf{0},1}} \chi_n(z)\chi_m(z)\, \frac{dz}{iz} \\ &= \frac{(-1)^{n+m+1} n! m!}{2\pi} \operatorname*{Res}_{z=0}(z^n - z^{-n})\left(z^{(m+1)} - z^{-(m+1)}\right) \\ &= \delta_{nm} \frac{(n!)^2}{\pi}. \end{aligned}$$

One might hope that there exists a smooth isotropic weight function $w \geq 0$, such that the set $\{H_s^{(n)}\}_{n \in \mathbb{N}\cup\{0\}}$ is orthogonal with respect to $\int_{-\infty}^{\infty} f(x) g(x)w(x)\, dx$, but this can not be the case as we will next show by contour-integration in the complex plane:



So, by letting $R \to \infty$ and $\delta \downarrow 0$ we have

$$\begin{aligned} &\int_{-\infty}^{\infty} H_s^{(n)} H_s^{(m)}(x) w(x^2+s^2)\, dx \\ &= \int_{is+\mathbb{R}} \chi_n(z)\chi_m(z) w(|z|^2)\, dz \\ &= \frac{1}{2} \operatorname*{Res}_{z=0}\left[z^{-n-m-2} - z^{-(n+1)}(z-2is)^{-(m+1)} \right. \\ &\quad \left. - z^{-(m+1)}(z-2is)^{-(n+1)} g(z)\right], \end{aligned}$$

with $g(z)$ analytic on $\{z \in \mathbb{C} : 0 \leq \Im(z) \leq s\} \setminus \{0\}$ such that $g(z) = w(z)$ on C_1.[24] From the fact that $\mathrm{Res}_{z=0} z^{-n} = \delta_{n1}$ it follows that either z^n, z^m, z^{2n+1}, or z^{2n+1} must appear in the expansion of $g(z)$ in order to obtain δ_{mn}. But this means that w will depend on n or m.

For each $n \in \mathbb{N} \setminus \{1\}$ the polynomial $P_{n,s}$, $s > 0$, has n real-valued zero's which lie around the zero's of $P_{n-1,s}$ (see Fig. 5).

We will show this by induction:

Since $P_{1,s}(x) = \frac{-2s}{\pi} x$ and $P_{2,s}(x) = \frac{s}{\pi}(-2s^2 + 6x^2)$, it is obvious that the above statement holds for $n = 2$.

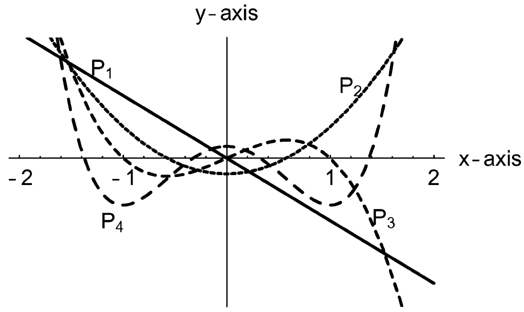


Figure 5. The graphs of $P_{n,1}$ for $n = 1, \dots, 4$.

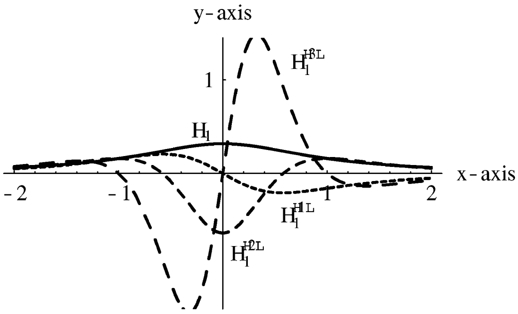


Figure 6. The graphs of $H_1^{(n)}$ for $n = 1, \dots, 4$.

Let $n \in \mathbb{N}$ and suppose that the above statement holds for this n. Then because $(x^2 + s^2) > 0$ and the fact that $P_{n+1,s}(x) = (x^2 + s^2)^{n+2} H_s^{n+1}(x) = (x^2 + s^2)^{n+2} \left(\frac{d}{dx}\right) H_s^n(x)$ it follows by Rolle's theorem that H_s^{n+1} has $n - 1 + 2 = n + 1$ real-valued zero's that lay around the n real valued zero's of $H_s^n(x)$.

Note that the zeros of the n-th order Poisson kernel derivative coincide with the zeros of $P_{n,s}$ and that the graphs show familiar behaviour as the Gaussian derivatives.

7.3. Minimizing Energies

Let f be an entire analytic vector with respect to the Gaussian resp. Poisson filtering, i.e. $f \in \mathcal{D}(\mathcal{A}^\infty) = \bigcap_{n \in \mathbb{N}} \mathcal{D}(\mathcal{A}^n) = \mathbb{H}_\infty$ such that $\sum_{k=1}^{\infty} \frac{s^n}{n!} \|\mathcal{A}^k f\|_{\mathbb{L}_2} < \infty$ for all $s > 0$. See appendix for definition and properties analytic vectors.

In Section 5.2.2 we have shown that the analytic vectors of the first order operators $\mathcal{A} = -\sqrt{-\Delta}$ and $\mathcal{A} = \partial$ are the same. The solution of the diffusion problem on the upper half space ($s > 0$) can now be written $u = e^{s\Delta} f$ and the solution of the Dirichlet problem on the upper half space ($s > 0$) can be written $u = e^{-s\sqrt{-\Delta}} f$. These solutions are (respectively) the solutions of the following minimization problems:

$$
\begin{aligned}
&\min_u \mathcal{E}_{\text{Gauss}}[u](s) \\
&\quad = \int_{\mathbb{R}^d} (f - u)^2 + \sum_{|\alpha|=1}^{\infty} \frac{s^{|\alpha|}}{\alpha!} (D^\alpha u)^2 \, d\mathbf{x}, \\
&\min_u \mathcal{E}_{\text{Poisson}}[u](s) \\
&\quad = \int_{\mathbb{R}^d} (f - u)^2 + \sum_{k=1}^{\infty} \frac{s^k}{k!} \|(-\Delta)^{k/4} u\|^2 \, d\mathbf{x} \quad s > 0.
\end{aligned}
$$

with $s > 0$ and $\alpha = \alpha_1 \dots \alpha_d$ multi-indices. Before we verify this statement, we remark that by using respectively partial integration and multinomial coefficients we have

$$
\begin{aligned}
&\sum_{|\alpha|=1}^{\infty} \frac{s^{|\alpha|}}{\alpha!} \|D^\alpha u\|^2 \\
&\quad = \sum_{k=1}^{\infty} s^k \sum_{|\alpha|=k} \frac{1}{\alpha_1! \alpha_2! \dots \alpha_d!} \\
&\qquad \times \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u, \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u \right)_{\mathbb{L}_2(\mathbb{R}^d)} \\
&\quad = \sum_{k=1}^{\infty} \frac{s^k (-1)^k}{k!} \\
&\qquad \times \left(\sum_{|\alpha|=k} \binom{k}{\alpha_1 \dots \alpha_d} \frac{\partial^{2\alpha_1}}{\partial x_1^{2\alpha_1}} \cdots \frac{\partial^{2\alpha_d}}{\partial x_d^{2\alpha_d}} u, u \right)_{\mathbb{L}_2(\mathbb{R}^d)} \\
&\quad = \sum_{k=1}^{\infty} \frac{s^k}{k!} ((-\Delta)^k u, u)_{\mathbb{L}_2(\mathbb{R}^d)}.
\end{aligned}
$$

With regard to the Poisson minimization we remark that $(-\Delta)^{k/4}$ is self adjoint. So we have

$$
\begin{aligned}
\mathcal{E}_{\text{Gauss}}[u](s) &= \|f - u\|^2_{\mathbb{L}_2(\mathbb{R}^d)} \\
&\quad + (u, e^{s\Delta} u)_{\mathbb{L}_2(\mathbb{R}^d)} - (u, u)_{\mathbb{L}_2(\mathbb{R}^d)} \\
\mathcal{E}_{\text{Poisson}}[u](s) &= \|f - u\|^2_{\mathbb{L}_2(\mathbb{R}^d)} \\
&\quad + (u, e^{-s\sqrt{-\Delta}} u)_{\mathbb{L}_2(\mathbb{R}^d)} - (u, u)_{\mathbb{L}_2(\mathbb{R}^d)},
\end{aligned}
$$

and therefore by using the Euler Lagrange principle:

$$
2(u - f) + 2 \sum_{k=1}^{\infty} \frac{s^k}{k!} (-\Delta)^k u = 0 \Leftrightarrow u = e^{s\Delta} f
$$

$$
2(u - f) + 2 \sum_{k=1}^{\infty} \frac{s^k}{k!} (\sqrt{-\Delta})^k u = 0 \Leftrightarrow u = e^{-s\sqrt{-\Delta}} f.
$$

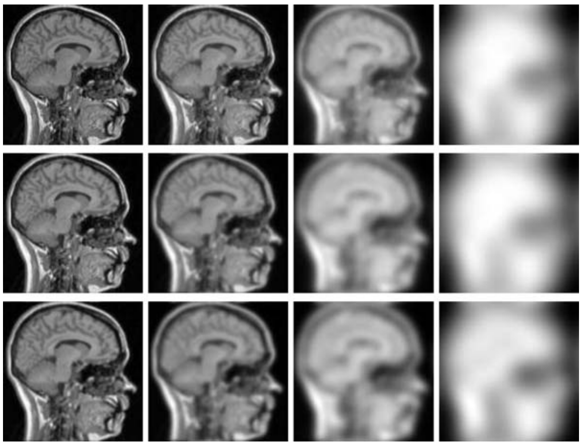


Figure 7. Various scale space representations of a 128×128 MR brain slice. Top row: $\alpha = \frac{1}{2}$ (Poisson scale space), middle row: $\alpha = \frac{3}{4}$, bottom row: $\alpha = 1$ (Gaussian scale space). The parameter α denotes the fractional power, cf. Eq. (66). Grey-values have been mapped to full range for the sake of clarity.

Note that because of Sobolev's lemma (see Yosida [39], pp. 174–175), the extra term added to the $\mathbb{L}_2$ (mean squared) part ensures that on every bounded open subset Ω of $\mathbb{R}^d$ there exists a $\tilde{u} \in C^\infty(\Omega)$ such that $u(\mathbf{x}) = \tilde{u}(\mathbf{x})$ almost everywhere on Ω.

8. Conclusion and Discussion

A unified framework to scale space theory on d-dimensional images on the unbounded domain is presented. An overcomplete set of scale space axioms leads uniquely to a parameterized class of scale spaces, the so-called α scale spaces, with corresponding filters of which the Fourier Transform is given by $e^{-s\|\omega\|^{2\alpha}}$. The special cases $\alpha = 1$ and $\alpha = 1/2$ lead to respectively Gaussian and Poisson scale space, with corresponding Gaussian and Poisson kernel. For implementation in the spatial domain Poisson filtering is a good alternative to Gaussian scale space in the sense that typical properties are maintained, besides the fact that all scale space axioms are satisfied. For implementation in the Fourier domain (which implicitly boils down to boundary conditions), cf. [5], all α scale spaces are relevant for practice.

The general theory of constructing a holomorphic semigroup with infinitesimal generator $-(-\mathcal{A})^\alpha$, $0 < \alpha < 1$ out of a strongly continuous semigroup with infinitesimal generator $\mathcal{A}$ that is a fractional power $0 < \alpha < 1$ of $-\mathcal{A}$ applied to the Laplace operator, reveals the *strong connection* between this whole family. Therefore it seems reasonable that Gaussian and Poisson scale space show similar results. Nevertheless, there are two main differences. First, the physical dimension of the Poisson evolution parameter is [LENGTH], while the physical dimension of the Gaussian evolution parameter equals [LENGTH^2]. Second, from an analytic point of view there is an essential difference: In Poisson scale space one solves the Dirichlet problem instead of the Diffusion problem on the upper half space, which means that the kernels and the filtered images are *harmonic functions*. This is a very nice property indeed, e.g. with regard to singularity analysis. The filtered images are then locally approximated by polynomials satisfying the PDE. In Poisson scale space this approximation will be a series of spherical harmonics converging uniformly on compact sets, with the useful property that the spherical harmonics form a complete orthogonal set on a ball or sphere. Further advantages over Gaussian scale space are: The mean value principle and its (Clifford) analytic extension, which corresponds ($d = 2$) to the monogenic scale space which is first introduced by Felsberg [9]. Although in the sense of grey-value flow, this $(d+1)$-dimensional vector scale space is analogue to the $(d+1)$-dimensional vector space consisting of the Gaussian scale space and its first order derivatives, it yields several local features which cannot be obtained in the latter.

Acknowledgments

Thanks to J. Boersma, professor at the department of Mathematics at the University of Technology Eindhoven for his useful remarks regarding the derivatives of the Poisson kernel. Thanks to dr.A.F.M. ter Elst at the department of Mathematics at the University of Technology Eindhoven for useful remarks concerning the theory of semigroups.

Special thanks to Michael Felsberg at the Computer Vision Laboratory of Linköping University, who worked on the special case of Poisson scale space (independently) cf. [7] and who brought to our attention that Koenderinks causality constraint had to be relaxed for Poisson scale space.

Appendix: Holomorphic Semigroups and Generalized Functions

Florack [11] first mentioned the role that test-functions and distributions play in scale space theory. On the one hand there is a state space, i.e. the space of all possible scalar field configurations and on the other hand there is a device space consisting of all admissible detector devices. It seems reasonable, from a physical point of view to describe a model for the detector device and to prescribe the state space by duality. Since the Gaussian kernels lie in the (Fourier invariant) Schwartz space $\mathcal{S}$, one can take it as a device space. As a result the state space is the set of all tempered distributions $\mathcal{S}'$. In [17] De Graaf has derived a theory of generalized functions based on holomorphic semi-groups. In this appendix we will apply little of this theory onto the Poisson semigroup.

In a Hilbert Space X we consider the evolution equation

$$\frac{du}{ds} = \mathcal{A}u$$

with $\mathcal{A}$ a negative unbounded self-adjoint operator. $\mathcal{A}$ is the infinitesimal generator of a holomorphic semi-group. Solutions $u(\cdot) : (0, \infty) \to X$ of this equation are called trajectories. Such a trajectory may or may not correspond to an "initial condition at $s = 0$" in X. The set of trajectories is considered as a space of generalized functions. The test function space is defined to be

$$\mathcal{S}_{X,\mathcal{A}} = \bigcup_{s>0} e^{s\mathcal{A}}(X).$$

Theorem 11. *Let X be a Banach space. Let Q be a strongly continuous, holomorphic semigroup, with infinitesimal generator $\mathcal{A} < 0$. Then $\mathcal{S}_{X,\mathcal{A}}$ consists exactly of those $f \in \mathcal{D}(\mathcal{A}^\infty)$ such that*

$$\sum_{k=1}^{\infty} \frac{s^k}{k!} \|\mathcal{A}^k f\| < \infty \quad \textit{for a certain } s > 0. \tag{79}$$

Proof: "$\subset$": Let $x \in \mathcal{S}_{X,\mathcal{A}} \Rightarrow \exists_{s>0} \exists_{x_s \in X}\, x = Q_s x_s$. Now since Q is holomorphic, there exists a $M > 0$ that $\|\mathcal{A}Q_s\| \leq M s^{-1}$, see (23). Consequently,

$$\begin{aligned} \|\mathcal{A}^n x\| &= \|(\mathcal{A}Q_{s/n})^n x_s\| \\ &\leq M^n (n/s)^n \|x_s\| \\ &\leq (eM/s)^n n! \|x_s\| \end{aligned}$$

where we have used the elementary inequality $n^n \leq e^n n!$. With these bounds it is now obvious that $\sum_{k=1}^{\infty} \frac{s^k}{k!} \|\mathcal{A}^k f\| < \infty$.

"$\supset$": Let $x \in \mathcal{D}(\mathcal{A}^\infty)$, such that (79) is convergent. Choose $t > 0$ such that $\sum_{n=1}^{\infty} \frac{s^n}{n!} \|\mathcal{A}^n x\| < \infty$ for all $s \leq 2t$. Since X is a Banach space, we have for $s \in (0, t)$

$$x_s = \sum_{k=1}^{\infty} \frac{s^k}{k!} (-\mathcal{A})^k x \quad \text{exists in } X,$$

(The sequence of partial sums is a Cauchy sequence). By differentiation we have $x = Q_s x_s = e^{s\mathcal{A}} x_s$. □

Remark. Elements of $\mathcal{D}(\mathcal{A}^\infty)$ that satisfy (79) are often called analytic vectors. If (79) holds for any $s > 0$ then then they are called *entire* analytic vectors.

Example: The Test Space Corresponding to Poisson Filtering

In this paragraph we will only observe $d = 1$.

The test function space which belongs to the Dirichlet problem (27) is given by $\mathcal{S}_{\mathbb{L}_2(\mathbb{R}^d), -\sqrt{-\Delta}}$, which—as we shall next show—equals a Gelfand-Shilov space, namely

$$\mathcal{S}_{\mathbb{L}_2(\mathbb{R}^d), -\sqrt{-\Delta}} = \mathcal{S}_1. \tag{80}$$

If the reader is not familiar with Gelfand-Shilov spaces he/she is referred to Gelfand and Shilov [15] (2nd edition, Chap. 4) for a general and profound theory with regard to their spaces. For our purposes we will only need $\mathcal{S}_1$, $\mathcal{S}^1$, $\mathcal{S}_{1,A}$, $\mathcal{S}^{1,A}$. First we give the definitions of these spaces and then we give a short overview of main results with references, about these Gelfand-Shilov spaces and finally we will show (80). Recall that the Schwartz space $\mathcal{S}$ consists of all $\phi \in C^\infty(\mathbb{R})$ such that,

$$\sup_{x \in \mathbb{R}} |x^k \phi^{(q)}(x)| \leq m_{qk} \quad \text{for certain } m_{qk} < \infty$$

with $(k, q = 1, 2, \ldots)$. To obtain Gelfand-Shilov spaces, we have to impose conditions on m_{qk}. Below, a summary of conditions on m_{qk} and corresponding Gelfand-Shilov spaces:

- $\mathcal{S}_1$: $m_{kq} = C_q A^k k^k$, with C_q, $A > 0$ depending on ϕ.
- $\mathcal{S}^1$: $m_{kq} = C_q B^q k^k$, with C_q, $B > 0$ depending on ϕ.
- $\mathcal{S}_{1,A}$: $m_{kq} = C_q (A \mp \delta)^k k^k$, with C_q depending on ϕ and *any* $\delta > 0$.

- $\mathcal{S}^{1,B}$: $m_{kq} = C_q(B \bar{+} \delta)^q k^k$, with C_q depending on ϕ and *any* $\delta > 0$.

The following relations hold:

- The test space $S_1 = \{\phi \in C^\infty(\mathbb{R}) : |\phi^{(q)}(x)| \leq C_q e^{-a|x|^{1/\alpha}}\}$, with $a = \frac{\alpha}{eA^{1/\alpha}}$ depending on ϕ. The idea behind this equality is that $\inf_{\xi\in\mathbb{R}} \frac{k^k}{|\xi|^k} = O(e^{-\frac{1}{e}|\xi|})$.
- The test space S^1 consists of all functions ϕ in $C^\infty(\mathbb{R})$ which can be extended analytically onto a strip $\{z \in \mathbb{C} : |\Im(z)| \leq h(\phi)\}$, which width $h(\phi)$ depends on ϕ. All members of S^1 are analytic, since the remainder $|\frac{h^q}{q!}\phi^{(q)}(x + \theta h)|$ of a Taylor expansion smaller than $\frac{|h|^q}{q!} C_0 B^q q^q$.[25]
- Fourier Transform $\mathcal{F}$ maps $\mathcal{S}^1$ onto $\mathcal{S}_1$ and visa versa.[26] Note that $\mathcal{F}^2$ is an isometrical mapping from $\mathcal{S}^1$ ($\mathcal{S}_1$) onto itself, since it maps $x \mapsto \phi(x)$ onto $x \mapsto \phi(-x)$.
- The space $\mathcal{S}_{1,A}$ consists exactly of those infinitely differentiable functions that satisfy

$$|\phi^{(q)}(x)| \leq C_{q\delta} e^{-(a-\delta)|x|} \quad \delta > 0$$

- The space $\mathcal{S}_1$ ($\mathcal{S}^1$) is the union of all $\mathcal{S}_{1,A}$ ($\mathcal{S}^{1,B}$), $A > 0$ ($B > 0$), i.e.

$$\mathcal{S}_1 = \bigcup_{A>0} \mathcal{S}_{1,A},$$
$$\mathcal{S}^1 = \bigcup_{B>0} \mathcal{S}^{1,B}.$$

  A sequence $\{\phi_n\} \subset \mathcal{S}_1$ converges to zero, if all ϕ_n belong to some $\mathcal{S}_{1,A}$ in which they converge to zero.

Note that we assumed functions to be infinitely differentiable. We will use analytic vectors f. In the semi-groups, we are interested in (Poisson and Gaussian) we have (respectively) $\mathcal{D}(\mathcal{A}^\infty) = \bigcap_{n=1}^\infty \mathbb{H}_n = \bigcap_{n=1}^\infty \mathbb{H}_{2n} = \mathbb{H}_\infty$. This space consists of all functions, whose generalized derivatives have finite $\mathbb{L}_2$-norms. To this end we notice that by Sobolevs lemma (Yosida [39] pp. 174–175) states that such functions are indeed almost everywhere infintely differntiable functions in the usual sense. So, we then arive at a special case of the above. For instance the space $\mathcal{S}^1$ will now consists of all infinitely differentiable functions f which can be extended analytically on a strip with widht h along the real axis in the complex plane such that

$$\sup_{-h<y<h} \int_{-\infty}^{\infty} |f(x + iy)|^2 \, \mathrm{dx} < \infty.$$

This space equals the space $\mathcal{S}_{\mathbb{L}_2(\mathbb{R}^d),-\sqrt{-\Delta}}$, since

$$\begin{aligned}
\mathcal{S}_{\mathbb{L}_2(\mathbb{R}^d),-\sqrt{-\Delta}} &= \mathcal{F}^{-1}\mathcal{F}\left(\bigcup_{s>0} e^{-s\mathcal{A}}(\mathbb{L}_2(\mathbb{R}^d))\right) \\
&= \mathcal{F}^{-1}\left(\bigcup_{s>0} e^{-s|\omega|}(\mathbb{L}_2(\mathbb{R}^d))\right) \\
&= \mathcal{F}^{-1}\left(\bigcup_{s>0} \mathcal{S}_{1,s}\right) \\
&= \mathcal{F}^{-1}\mathcal{S}_1 = \mathcal{S}^1.
\end{aligned}$$

Notes

1. Following Pauwels we dismiss separability as a viable basic axiom, since it is a coordinate dependent notion. It is a straightforward that any C^1, normalised filter which is both separable and isotropic must be a Gaussian, see (69).
2. The usual causality principle must be relaxed, but the main result remains.
3. Scale space theory can be approached in three ways: as semigroups, as evolution equations (partial differential equations) and as stochastic processes (probability theory). This article handles with the first two approaches, but the connection with probability theory will be further examined in future work.
4. Formally, the Laplace Transform maps the space $\mathbb{L}_2(\mathbb{R}^+)$ onto the Hardy Lebesgue class $H^2(0)$, cf. [39] p. 163.
5. To every $f \in \mathbb{L}_1(\mathbb{R}^d)$ corresponds a distribution Λ_f. Note that the definitions of Fourier transform of resp. f and Λ_f coincide (as they should): $(\hat{\Lambda_f})(\phi) = \Lambda_f(\hat{\phi}) = \int f\hat{\phi} = \int \hat{f}\phi = (\Lambda_{\hat{f}})(\phi)$. Moreover, the same argument is valid if $f \in \mathbb{L}_2(\mathbb{R}^d)$.
6. Using Cauchy Schwarz and the fact that $u \to f$ for $s \downarrow 0$ in $\mathbb{L}_2$ sense one obtains that $\mathcal{E}(u)$ is also continuous in 0.
7. In a Hilbert space the Riesz identification identifies the original space with its dual.
8. This coincides for the case $\alpha = 1/2$ (Poisson semigroup) with a general result on strongly elliptic forms. See Robinson [32] Proposition 3.6.
9. This boundary condition ensures that the average grey value is maintained. Analogue to thermic isolation in heat physics.
10. One can show that such a limit always exist.
11. For the constant α in part 2. of (23) in the Poisson case, see proof of Theorem 10.
12. Moreover if a function satisfies the mean value principle, then it must be harmonic!
13. Notice that this function cannot be a scale space function.
14. $f(\mathbf{a}) = \frac{1}{\tau_n R^n} \int_{B_{\mathbf{a},R}} f(\mathbf{x})\, d\mathbf{x}, \mathbf{a} \in \Omega, R > 0$.
15. It is shown in Robinson [32] p. 49 that the Hilbert transform is unbounded as an operator from $\mathbb{L}_\infty(\mathbb{R})$.
16. Physically, one could say that the potential field of a positive charge and that of a mirrored negative charge cancel out along $s = 0$.
17. It is sloppy to use $\Delta S(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = -\delta(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ and apply Greens second identity on $\Omega = \mathbb{R}^d \times \mathbb{R}^+$, but interesting enough the result is the same.

18. If restricted to the subspace of analytic signals the Cauchy operator is an isometric isomorphism such that the non tangential limit $\lim_{s\downarrow 0} Cg(\cdot, s) = g(\cdot)$ for all $g \in H^2(\partial\mathbb{C}_+)$, cf. [16] p. 113.
19. Actually, Sufficiently smooth should be replaced by analytic vectors, see Appendix, but in order to avoid confusion with analytic function this term is avoided.
20. In general inverse Laplace transformation can be done by using the inversion formula
$$f(\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda z} [\mathcal{L}(f)](z) \, \mathrm{d}z$$
with σ such that $\mathcal{L}(f)(z)$ is analytic on $\Re(z) \geq \sigma$. So, $q_{s,\alpha}$ is in fact the inverse Laplace transform of $\mu \mapsto e^{-s\mu^\alpha}$.
21. Equation (52) corresponds to the scale invariance axiom, Axiom 3.
22. One might think of using the Campbell-Baker-Hausdorff formula, since the clifford product commutator is known.
23. Formally, we have $s > 0$ so Ψ_n is only defined for $s > 0$. Therefore, use Schwarz reflection principle if necessary.
24. Substitute $(z - 2is)$ for $\bar{z}$ in the expansion of $w(|z|^2)$ with respect to $|z|^2 = z\bar{z}$.
25. From Stirling's Formula $q! = q^{q+(1/2)} e^{-q} \sqrt{2\pi} E_q \quad (E_q \to 1)$ it follows that $h(q) = \frac{1}{B(\phi)e}$.
26. In general we have $\mathcal{F}(\mathcal{S}_\alpha^\beta) = \mathcal{F}(\mathcal{S}_\beta^\alpha)$, so the Fourier invariant spaces are $\mathcal{S}_\alpha^\alpha$. For example the Schwarz-Space, $\alpha = \infty$.

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