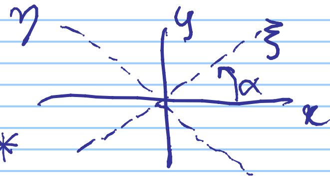
Note Title

Image Structure   Friday Oct. 24 2003

24-10-2003



Diff. Inv.

Example 1: $\|\nabla u\|^2 = u_x^2 + u_y^2 = *$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

$$* = \left(\frac{\partial \xi}{\partial x} u_\xi + \frac{\partial \eta}{\partial x} u_\eta\right)^2 + \left(\frac{\partial \xi}{\partial y} u_\xi + \frac{\partial \eta}{\partial y} u_\eta\right)^2 = (c u_\xi + s u_\eta)^2 + (-s u_\xi + c u_\eta)^2 =$$

$$= (c^2+s^2) u_\xi^2 + (2cs - 2cs) u_\xi u_\eta + (c^2+s^2) u_\eta^2 = u_\xi^2 + u_\eta^2$$

Example 2 :

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Ex 2: $\Delta u = u_{xx} + u_{yy} = \left(\frac{\partial \xi}{\partial x}\right)^2 u_{\xi\xi} + 2\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} u_{\xi\eta} + \left(\frac{\partial \eta}{\partial x}\right)^2 u_{\eta\eta} +$

$(x \longleftrightarrow y)$

$= (c^2 + s^2)\, u_{\xi\xi} + (2cs - 2cs)\, u_{\xi\eta} + (c^2 + s^2)\, u_{\eta\eta} =$

$= u_{\xi\xi} + u_{\eta\eta}$

$$\|\nabla u\|^2 = \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^i} \overset{\text{def}}{=} \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^i} = u_i u_i = *$$

$$\underset{\text{in } n=2}{R =} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \qquad RR^T = I = R^T R$$

$$\vec{\xi} = R\vec{x} \quad \text{or} \quad \xi_i = R_{ij} x_j$$

$$\frac{\partial \xi^i}{\partial x^j} = R_{ij} \qquad R_{ik}R_{jk} = \delta_{ij} = R_{ki}R_{kj}$$

$$* = \frac{\partial \xi^j}{\partial x^i}\frac{\partial u}{\partial \xi^j}\frac{\partial \xi^k}{\partial x^i}\frac{\partial u}{\partial \xi^k} = \underbrace{R_{ji}\frac{\partial u}{\partial \xi^j} R_{ki}}_{\delta_{jk}}\frac{\partial u}{\partial \xi^k} = \frac{\partial u}{\partial \xi^k}\frac{\partial u}{\partial \xi^k}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^i \partial x^i} = \frac{\partial \xi^k}{\partial x^i} \frac{\partial}{\partial \xi^k}\left(\frac{\partial \xi^j}{\partial x^i} \frac{\partial u}{\partial \xi^j} \right) = \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^j}{\partial x^i} \frac{\partial^2 u}{\partial \xi^k \partial \xi^j} =$$

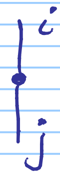
$$= R_{ki} R_{ji} \frac{\partial^2 u}{\partial \xi^k \partial \xi^j} = \delta_{kj} \frac{\partial^2 u}{\partial \xi^k \partial \xi^j} = \frac{\partial^2 u}{\partial \xi^k \partial \xi^k}$$

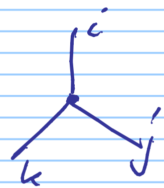
Cond. $\frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^i}$ or $\frac{\partial^2 A}{\partial x^i \partial x^i}$ diff. inv. if A/B are diff. inv.

Generalisation: $u \mathrel{\hat{=}}$ 

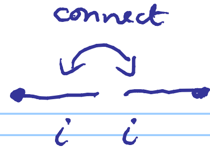
$\dfrac{\partial u}{\partial x^i} \mathrel{\hat{=}}$ 

$i = 1, \dots, n$

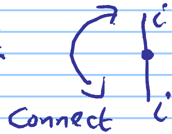
$\dfrac{\partial^2 u}{\partial x^i \partial x^j} \mathrel{\hat{=}}$ 

$\dfrac{\partial^3 u}{\partial x^i \partial x^j \partial x^k} \mathrel{\hat{=}}$ 

etc.

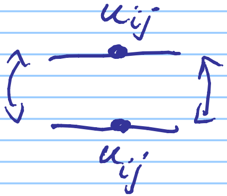
$\|\nabla u\|^2 = \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^i} \hat{=}$ 

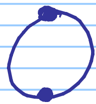
$=$ 

$\Delta u = \frac{\partial^2 u}{\partial x^i \partial x^i} \hat{=}$ 

$=$ 

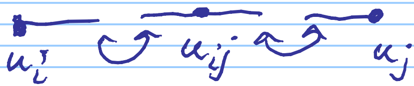
Other diff. inv's:



$=$  $\hat{=} \; u_{ij}u_{ij}$

$n=2$: $u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2$

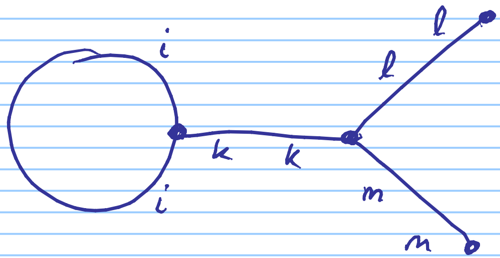
$n=3$: $u_{xx}^2 + u_{yy}^2 + u_{zz}^2 + 2(u_{xy}^2 + u_{xz}^2 + u_{yz}^2)$



$=$  $= u_i u_{ij} u_j$ in $n=2$:

$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}$

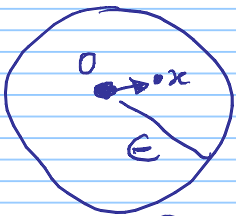
One more diff. inv:



$$\hat{=} \quad u_{iik}\, u_{klm}\, u_{l}\, u_{m} =$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial^3 u}{\partial x^i \partial x^i \partial x^k} \frac{\partial^3 u}{\partial x^k \partial x^l \partial x^m} \frac{\partial u}{\partial x^l} \frac{\partial u}{\partial x^m}$$

• crit. point: $\frac{\partial u}{\partial x^i} = 0$ (at origin)



$$I_\epsilon \overset{\text{def}}{=} \int_{\Omega_\epsilon} \|\nabla u(x)\|^2 \, dV \qquad \ast$$

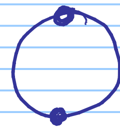
$$\Omega_\epsilon : \|x\|^2 \leq \epsilon$$

$$\|\nabla u(x)\|^2 = u_i(x)\, u_i(x) = \{u_i + u_{ij} x^j + h.o.t.\}\{u_i + u_{ik} x^k + h.o.t.\} =$$

$$= u_{ij} u_{ik} x^j x^k + \dots$$ subst. in $\ast$ yields:

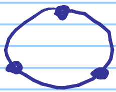
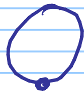
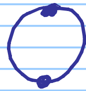
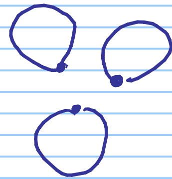
$$I_\epsilon = u_{ij} u_{ik} \int_{\Omega_\epsilon} x^j x^k \, dV = K_\epsilon \delta^{jk} u_{ij} u_{ik} \quad \text{some } K_\epsilon > 0$$

$$= K_\epsilon u_{ij} u_{ij}$$

Take e.g. $\displaystyle \lim_{\epsilon \downarrow 0} \frac{I_\epsilon}{K_\epsilon} = u_{ij} u_{ij} \hat{=}$ 

$$\| \quad \text{trace } H^2$$

Redundancies: $n \overset{!}{=} 2$

 $= \frac{3}{2}$   $- \frac{1}{2}$ 

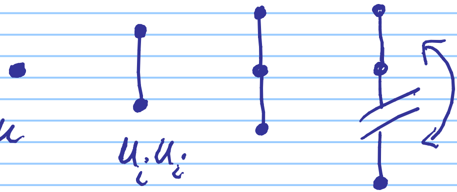
$$u_{ij}\, u_{jk}\, u_{ki} \overset{*}{=} \frac{3}{2}\, u_{ii}\, u_{jk}\, u_{kj} - \frac{1}{2}\, u_{ii}\, u_{jj}\, u_{kk}$$

=

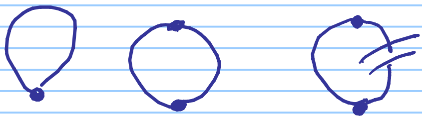
assume
(x,y)-coord. system is such that $\left(u_{ij}\right)_{\substack{i=1,2\\ j=1,2}} = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$*$: $$a^3 + b^3 = \frac{3}{2}(a+b)(a^2+b^2) - \frac{1}{2}(a+b)^3 \quad \forall a, b$$

Complete set of 2^{nd} order diff. inv.'s in n dim's: $(n \geq 2)$



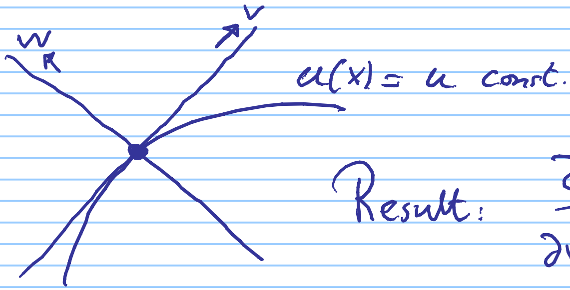
u $\qquad u_i u_i \qquad u_i u_{ij} u_j$



$u_{ii} \qquad u_{ij} u_{ji}$

Alternative way to construct diff. inv's including "relative" invariants (invariants up to mirroring of coord' axes):

$n=2$



Result: $\dfrac{\partial^{i+j} u}{\partial v^i \partial w^j}$ is an absolute or relative (i odd) diff. invariant

Relation to arbitrary global Cartesian coord. syst:

$$\frac{\partial}{\partial v} = V_i \frac{\partial}{\partial x^i} \qquad V_i = \frac{\epsilon_{ij} u_j}{\|\nabla u\|}$$

$$\frac{\partial}{\partial w} = W_i \frac{\partial}{\partial x^i} \qquad W_i = \frac{u_i}{\|\nabla u\|}$$

$$\epsilon_{ij} = \begin{cases} 1 & \text{if } i=1, j=2 \\ -1 & \text{if } i=2, j=1 \\ 0 & \text{if } i=j \end{cases}$$

const ∂

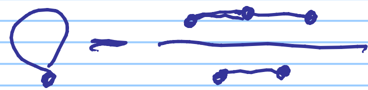
Ex.
$$\frac{\partial^2 u}{\partial v^2} = \frac{\epsilon_{ij} u_j}{\|\nabla u\|} \frac{\partial}{\partial x^i} \frac{\epsilon_{kl} u_l}{\|\nabla u\|} \frac{\partial}{\partial x^k} =$$

const.

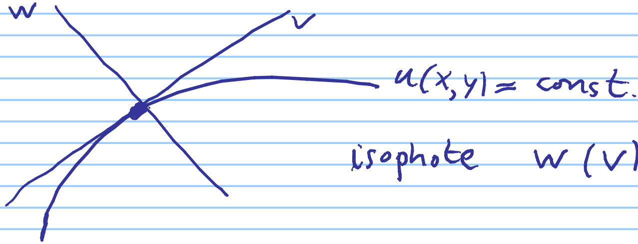
$$= \frac{\epsilon_{ij}\epsilon_{kl} u_j u_l}{\|\nabla u\|^2} \frac{\partial^2 u}{\partial x^i \partial x^k} \quad \Longrightarrow$$

Lemma: $\epsilon_{ij}\epsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$

$$\frac{\partial^2 u}{\partial v^2} = u_{ii} - \frac{u_j u_l u_{jl}}{\|\nabla u\|^2} =$$



Ex. isophote curvature $n=2$:



$$u(v, w(v)) = u_0 \quad \text{so} \quad \frac{d}{dv} u(v, w(v)) = 0$$

$$\frac{\partial u}{\partial v} + w'(v)\frac{\partial u}{\partial w} = 0 \quad \text{so at } v=0 \text{ where } w'(0) = 0 \text{ so } u_v = 0$$

$\frac{d}{dv}$:

$$\frac{\partial^2 u}{\partial v^2} + 2w'(v)\frac{\partial^2 u}{\partial v \partial w} + w''(v)\frac{\partial u}{\partial w} + w'(v)^2\frac{\partial^2 u}{\partial w^2} = 0 \quad \text{so at } v=0$$

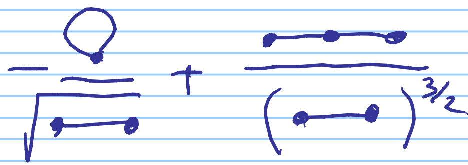
$$\kappa \equiv w''(0) = -\frac{u_{vv}}{u_w}$$

Cond.

$$K = -\frac{u_{vv}}{u_{ww}} = -\frac{\epsilon_{ij} u_j \epsilon_{kl} u_l \frac{\partial^2 u}{\partial x^i \partial x^k}}{\|\nabla u\| u_m \frac{\partial u}{\partial x^m}} =$$

$$= -\frac{\epsilon_{ij}\epsilon_{kl} u_j u_l u_{ik}}{\|\nabla u\|^3} = +\frac{[-\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] u_j u_l u_{ik}}{\|\nabla u\|^3} =$$

$$= -\frac{\Delta u}{\|\nabla u\|} + \frac{u_i u_j u_{ij}}{\|\nabla u\|^3} =$$



$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = u_w^2 u_{ww} =$$

$$\left(\frac{\partial}{\partial w} = \frac{u_i}{\|\nabla u\|} \frac{\partial}{\partial x^i} \right) = \left(\frac{u_i}{\|\nabla u\|} \frac{\partial u}{\partial x^i} \right) \left(\frac{u_j}{\|\nabla u\|} \frac{\partial u}{\partial x^j} \right) \frac{u_k u_\ell}{\|\nabla u\|^2} \frac{\partial^2 u}{\partial x^k \partial x^\ell}$$

$$= u_k u_\ell u_{k\ell} =$$ 

THE END