HOMEWORK ASSIGNMENT IMAGE STRUCTURE

Course code: SA5. Date: October/November 2003.

Read this first

- Group assignments:
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- Format: Each group hands in one electronic document in either PostScript, PDF, Mathematica, or Word.
- Deadline: The electronic document should be received by email at `L.M.J.Florack@tue.nl` no later than November 20, 2003.

1. *Stability of critical points in gradient magnitude image.* Consider the $n+1$-dimensional scale space gradient magnitude image $v(x,t) = \|\nabla u(x,t)\|^2$, in which $u(x,t)$ is obtained by convolution of the (n-dimensional) high resolution raw image, $f(x)$ say, with a normalized Gaussian

$$\phi_t(x) = \frac{1}{\sqrt{4\pi t}^{\,n}} \exp\left[-\frac{\|x\|^2}{4t}\right].$$

Note that this means that $u(x,t)$ (but not $v(x,t)$) satisfies the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t)\,,$$

with initial condition $u(x,0) = f(x)$.

a. It is tacitly assumed that the raw image is defined on all of $\mathbb{R}^n$ by "zero padding". Solve the heat equation subject to the given initial condition (and suitable boundary condition at infinity) so as to prove that the solution is indeed given by the convolution

$$u(x,t) = (f * \phi_t)(x)\,.$$

(Hint: Fourier transformation.)

b. Show that $\nabla u(x,t) = (f * \nabla\phi_t)(x)$. Sketch the proof of well-posedness of this differentiation method, i.e. argue that a small perturbation of raw data $\delta f(x)$ induces a small perturbation $\delta\nabla u(x,t)$ for each fixed t. What does "small" mean here?

c. Consider a *spatial critical point* of $v(x,t)$, i.e. a point at which the n first order spatial derivatives of $v(x,t)$ vanish for some fixed scale t. Is such a point also a spatial critical point of $u(x,t)$? Argue that there are two qualitatively distinct types of spatial critical points of $v(x,t)$.

d. Compute the lowest order nontrivial Taylor polynomial of $v(x, t)$ *at a spatial critical point*, the origin say, in terms of spatial derivatives of u at the origin $(0, 0) \in \mathbb{R}^{n+1}$ (assuming that both $\|x\|$ as well as $|t|$ are small).

e. Use this to evaluate the following "total variation norm" by approximation in terms of spatial derivatives of u at the origin:

$$\mathrm{TV}_\epsilon[v] \stackrel{\text{def}}{=} \int_{\Omega_\epsilon} \|\nabla v(x, t)\|^2 \, dV \,,$$

in which $\Omega_\epsilon : \|x\|^2 < \epsilon^2$ for suitably small $\epsilon > 0$, i.e. a spatial ball of radius ϵ. In your answer you should argue that

$$\int_{\Omega_\epsilon} x_i \, x_j \, dV = K_\epsilon \, \delta_{ij} \quad (i, j = 1, \ldots, n) \,,$$

for some ϵ-dependent constant K_ϵ with $\lim_{\epsilon \to 0} K_\epsilon = 0$, but you don't need to evaluate the numerical value of this constant.

f. Define a differential invariant at the origin which captures the total variation norm locally by carrying out a suitable limiting procedure for the above integral as $\epsilon \to 0$.

g. Give a diagrammatic representation of the differential invariant derived above which holds in any dimension. If possible, derive an equivalent diagrammatic representation in terms of "irreducible diagrams" for the special cases $n = 2$ and $n = 3$ (cf. [2, Section 5.3.2]).

h. Give an equivalent formula in terms of the "Hessian gauge", i.e. in terms of a local spatial coordinate frame at the origin, with coordinates $p = (p_1, \ldots, p_n)$ say, such that all mixed second order partial derivatives of u in this frame vanish at the origin.

g. Argue why this differential invariant can (or cannot) be used as a measure of stability of the critical point of interest.

2. *Beyond Cartesian invariance.*

Consider the following p-parametrised differential invariant in $n = 2$ dimensions, with $p \geq 0$:

$$I_p(u) \stackrel{\text{def}}{=} \frac{\partial^2 u}{\partial v^2} \left[\frac{\partial u}{\partial w} \right]^{p-1} ,$$

in which (v, w) are the coordinates relative to a positively oriented, local Cartesian coordinate frame at an implicitly given point in the image, the origin say, such that the positive w-axis is aligned with the gradient (the "gradient gauge"). Thus the (v, w)-frame varies from point to point in the image. We will henceforth write $I_p = u_{vv} \, u_w^{p-1}$ for short.

a. Express I_p in terms of an arbitrary, global Cartesian coordinate system (x, y) in terms of tensor index notation using the Einstein summation convention. That is, your expression should contain only full contractions of index pairs indicating partial derivatives.

b. Express I_p in terms of an arbitrary, global Cartesian coordinate system (x, y) in terms of explicit x- and y-derivatives.

c. Give the corresponding diagrammatic representation of I_p (cf. [2, Section 5.3] for examples).

d. Argue why I_p is a Cartesian differential invariant. To which group does the notion of "invariance" refer?

e. Show that I_0 is invariant under the group of invertible grey-level transformations, i.e. if $v = \gamma(u)$ with $\gamma'(u) > 0$ for all grey-levels u (e.g. histogram equalization), and if we define

$$\gamma^* I_0(u) \stackrel{\text{def}}{=} I_0(\gamma(u)) ,$$

then $\gamma^* I_0 = I_0$. Show that this is no longer true if $p \neq 0$.

f. The invariance property above indicates that I_0 must refer to some local geometric property in the image which does not depend on a grey-level metric. Which?

g. Now consider I_3. Show that this invariant is well-defined even if the (v, w)-gauge does not exist, i.e. if the point of interest happens to be a critical point ($u_w = 0$). Is this also true for I_p in general?

h. Show that I_3 is invariant under the group of area preserving affine transformations, i.e. transformations of the type $x'_i = a_{ij}\, x_j + b_i$ with $\det a = \pm 1$. (Note: You only need to verify invariance under the non-Cartesian extension, i.e. area preserving affine scalings.) Show that this generalised invariance no longer holds if $p \neq 3$.

i. I_3 has been proposed as a viable "corner detector" [1]. A "corner" is defined as any point at which the gradient magnitude ("edgeness") and the curvature of the local iso-intensity contour are simultaneously large. (Thus it would be better to talk about "cornerness", since it is essentially undefined what "large" means and since I_3 can be computed at any point in the image.) Argue what the abovementioned invariance implies for the interpretation of what a "corner of given strength" is. (In your answer you should take into account what the precise trade-off is between the contributions of "edgeness" and curvature.)

j. A popular "edge" detection method is obtained by taking the zero-crossings of the Laplacian, i.e. the loci of points where $\Delta u = 0$. In general this leads to closed contours, or contours that end on the image boundary. However, this method is known to produce counter-intuitive edges at sharp corner points. Explain the failure of the Laplacian zero-crossings method at such point by showing that $\Delta u = u_{ww} - \kappa\, u_w$, with $\kappa = -u_{vv}/u_w = -I_0$, and interpreting the right hand side of this formula in terms of local image features (especially near and off "corners" and "edges").

Referenties

[1] J. Blom. *Topological and Geometrical Aspects of Image Structure*. PhD thesis, University of Utrecht, Department of Medical and Physiological Physics, Utrecht, The Netherlands, 1992.

[2] L. M. J. Florack. *Image Structure*, volume 10 of *Computational Imaging and Vision Series*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.